

Chapter 8 Application Project: A Sturdy Foundation

On November 7, 1940, the original Tacoma Narrows Bridge spectacularly collapsed under the sustained effect of strong and rhythmic wind forces. This stunning disaster was the result of what was then a poorly studied phenomenon called *aeroelastic flutter* caused by undamped periodic forces; its effect is closely related to what is called *forced mechanical resonance*.

Resonance might happen when a periodic external force is acting on an oscillating system. In this project, we will examine some conditions under which the phenomenon might occur, with the assumption that no damping forces are present. Such motion is called *forced undamped motion*.

1. The following initial value problem represents a spring-mass system where the oscillating mass m is acted upon by an external force as well as the restoring force of the spring it is attached to. In addition to k (the usual spring constant), f is also a constant.

$$m \frac{d^2 y}{dt^2} + ky = f \sin(\theta t); \quad y(0) = 0; \quad y'(0) = 0$$

- a. Compare the above IVP to that in Example 4 of Section 8.4 and describe any differences. Relate any mathematical differences to the forces acting on the oscillating mass.
 - b. In words, describe why you would expect a major difference in the motion of a spring-mass system described by the above IVP, in contrast with Example 4 of Section 8.4. Are there any damping forces present?
 - c. Find the period and frequency of the expression on the right-hand side of the differential equation.
2.
 - a. Find the general solution $y_c(t)$ of the associated homogeneous equation in the IVP above. Use the conventional notation $\omega = \sqrt{k/m}$.
 - b. Starting with $y_p(t) = A \cos(\theta t) + B \sin(\theta t)$ as the initial guess, find a particular solution y_p of the differential equation in Question 1. (**Hint:** See Exercises 39–44 and the preceding discussion in Section 8.4.)
 - c. Use your answers to parts a. and b. as well as the initial conditions to find the solution $y(t)$ to the initial value problem of Question 1.
 3. Find the formula for $Y(t) = \lim_{\theta \rightarrow \omega} y(t)$. (**Hint:** Use L'Hôpital's Rule.)
 4.
 - a. Use the formula for $Y(t)$ that you obtained in Question 3 to examine $\lim_{t \rightarrow \infty} |Y(t)|$, and describe what happens to the amplitude of the oscillations when $\theta = \omega$ and t increases without bound.
 - b. Use technology to obtain a graph that illustrates the behavior of $Y(t)$ as $t \rightarrow \infty$. (Choose your own values for the unspecified parameters. Answers will vary.)
 - c. Use your answer to Question 4a (along with a limit argument) to explain the physical effect of θ approaching ω in this type of forced, undamped oscillating motion.
 5. Your work on Question 4 and the answers you found provide mathematical insights into the physical phenomenon called *resonance*. This will occur in lightly damped or undamped systems when the frequency of the external driving force approaches the oscillating system's natural frequency, that is, when $\theta \rightarrow \omega$. Use your results to write a short paragraph explaining this phenomenon.
 - 6.* There is actually a direct way to obtain $Y(t)$, that is, by finding the solution of the following initial value problem.

$$m \frac{d^2 y}{dt^2} + ky = f \sin\left(\sqrt{\frac{k}{m}} t\right); \quad y(0) = 0; \quad y'(0) = 0$$

Find the solution of the IVP above to verify the formula for $Y(t)$ you obtained in Question 3b.