Chapter 7 Conceptual Project: Infinite Wisdom

In this project, we will derive a famous infinite product named after its discoverer, the English mathematician John Wallis (1616–1703). Wallis introduced the symbol ∞ for infinity, and in turn he used $1/\infty$ to denote an *infinitesimal* quantity. He contributed to the development of *infinitesimal calculus* (it wasn't until the 19^{th} century that infinitesimals were replaced by limits in the works of Bolzano, Cauchy, and Weierstrass).

1. For a nonnegative integer n, let

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx.$$

Find I_0, I_1, I_2, and I_3.

2. Show that if $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(**Hint:** See Exercise 81 of Section 7.1.)

3. Use Questions 1 and 2 to find I_4, I_5, I_6, and I_7.

4. Show that in general,

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

while

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \ldots \cdot \frac{2}{3}.$$

(**Hint:** Observe a pattern or use induction.)

5. Use Question 4 to show that

$$\frac{I_{2n}}{I_{2n+1}} = \frac{3^2 5^2 \cdot \ldots \cdot (2n-1)^2 (2n+1)}{2^2 4^2 \cdot \ldots \cdot (2n)^2} \cdot \frac{\pi}{2}$$

holds for all n.

6. Show that

$$\frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n}.$$

7. Prove the inequalities

$$I_{2n-1} \geq I_{2n} \geq I_{2n+1}.$$

(**Hint:** Use the definition of I_n from Question 1 and compare the integrands.)

8. Use Questions 6 and 7 to show that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n},$$

and use this observation to prove that

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

9. Use your answers to the previous questions to derive **Wallis' product**, as follows.

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^2 4^2 \cdot \ldots \cdot (2n)^2}{3^2 5^2 \cdot \ldots \cdot (2n-1)^2 (2n+1)}$$