Chapter 6 Conceptual Project: A Frictionless Flight

In this project, we will expand upon our explorations from Exercises 48 through 55 of Section 6.5 (also see the discussion preceding those exercises). In particular, we will determine an equation satisfied by the velocity of a projectile launched with initial velocity v_0, taking into consideration that acceleration caused by gravity decreases with altitude. (This is important when objects are launched to great altitudes.) We will then use our equation to find the maximum height attained by the projectile. This will lead us to the value of the so-called *escape velocity*, the velocity needed for an object to be able to overcome Earth's gravitational field without further propulsion. (In turn, since gravity is conservative, this is the same velocity an object would achieve if pulled in by gravity from an "infinite distance.") We are ignoring all retarding forces (such as air resistance or friction) in this discussion.

1. Recall from Exercise 48 of Section 6.5 that g, the acceleration caused by gravity on a free-falling mass near Earth's surface, is approximately

$$g = \frac{MG}{R^2},$$

where M and R are the mass and radius of Earth, respectively, and G is the universal gravitational constant. However, a launched projectile's acceleration caused by gravity is negative (if we are assuming the positive direction is upward) and actually depends on its height h above Earth's surface. In particular, use Newton's Law of Gravitation to show that this dependence is given by the equation

$$a(h) = \frac{-gR^2}{(R+h)^2},$$

where $a(0) = -g$, as we would expect. (Actually, $a(h) \approx -g$ when h is negligible compared to Earth's radius.)

2. Show that if $v = v(h)$ denotes the velocity of the projectile, then

$$\frac{d}{dh}\left(v^2\right) = 2\frac{dv}{dt}.$$

(**Hint:** Use the Chain Rule.)

3. Use the above results to show that

$$\frac{d\left(v^2\right)}{dh} = \frac{-2gR^2}{(R+h)^2}.$$

4. Integrating both sides of the preceding equation with respect to h, show that $v = v(h)$ satisfies the equation

$$v^2 = v_0^2 - 2gR\left(1 - \frac{R}{R+h}\right).$$

(**Hint:** After integrating, use the fact that $v(0) = v_0$.)

5. Use the equation found in Question 4 to find the maximum height attained by the projectile. (**Hint:** Use the fact that $v = 0$ when the projectile reaches its maximum height.)

6. Find a formula for the escape velocity v_e of the projectile; then use the data found in the exercises of Section 6.5 (Exercises 48–55 and the preceding discussion) to express your answer in kilometers per second. (**Hint:** Use the fact that if $v_0 = v_e$, the projectile will "travel to infinity.")

7. Find the escape velocity of the projectile if it is launched on the moon. (**Hint:** For moon data, see Exercise 53 of Section 6.5.)