Chapter 15 Application Project: Dido's Clever Circle

Geometric inequalities abound in mathematics, and their history stretches back thousands of years. One of the earliest is the so-called *Isoperimetric Inequality*, which, in two dimensions, states that among all planar regions with a given fixed perimeter, a circle encloses the maximum possible area. This is also known as Dido's problem, after the tale in Virgil's *Aeneid* that recounts Dido's cleverness in bargaining for land on which to found the city of Carthage.

A related inequality is the following, which we first present in the setting of $\mathbb{R}^3$. Let D be an open region of $\mathbb{R}^3$ which is contained in a ball B_r of radius r. Let S denote the surface of D, and assume that S is smooth. Let V denote the volume of D and let A denote the area of S (that is, $V = \iiint\limits_D dV$ and $A = \iint\limits_S d\sigma$). Then

$$V \le \left(\frac{r}{3}\right) A.$$

In this project, you will first use the Divergence Theorem to prove the above inequality and then see how the same proof allows the inequality to be generalized to n-dimensional space. Finally, an alternative proof that illustrates the connection to the Isoperimetric Inequality will be outlined.

1. First, note that we can assume D and B_r are positioned so that B_r is centered at the origin (since V and A are unchanged by moving D and B_r we can just move them so that B_r is centered). Let $\mathbf{r} = \langle x, y, z \rangle$ and define $\mathbf{F}(\mathbf{r}) = \mathbf{r}$. Determine $\nabla \cdot \mathbf{F}$.

2. Let $\mathbf{n}$ be the outward-pointing field of unit vectors normal to S. Show that $|\mathbf{F} \cdot \mathbf{n}| \le r$ for all such $\mathbf{n}$. (**Hint:** Consider the Cauchy-Schwarz Inequality of Section 11.3 Exercise 69.)

3. Calculate $\iiint\limits_D \nabla \cdot \mathbf{F}\, dV$ exactly and use your result from Question 2 to determine an upper bound for $\left| \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \right|$. Use the Divergence Theorem to relate your results and arrive at the inequality

$$V \le \left(\frac{r}{3}\right) A.$$

4. As mentioned in the introduction of Topic 1 of Section 15.8, the Divergence Theorem is true in any number of dimensions. If we consider D to be an open region of $\mathbb{R}^n$, where $n \ge 2$, and if we let ∂D denote the boundary of D, then it is still the case that

$$\int\limits_{\partial D} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\limits_D \nabla \cdot \mathbf{F}\, dV,$$

where the integration takes place in $\mathbb{R}^{n-1}$ over ∂D and in $\mathbb{R}^n$ over D. In $\mathbb{R}^n$, B_r denotes the n-dimensional ball of radius r, which in words can be defined as all the elements of $\mathbb{R}^n$ that lie within r units of the origin (we will again assume B_r is centered at the origin). With the understanding that $V = \int\limits_D dV$ and $A = \int\limits_{\partial D} d\sigma$, replicate your work in Questions 1 through 3 (with slight modifications) to show that if D is contained in B_r, then

$$V \le \left(\frac{r}{n}\right) A.$$

5. To gain a better understanding of how the inequality works, consider how it applies to shapes centered at the origin in $\mathbb{R}^2$ (that is, consider the case $n = 2$) Note that in $\mathbb{R}^2$, V (which is the volume of D) actually corresponds to the area of D, and A (which is the surface area of the boundary of D) is the length of the perimeter of D. For instance, if D is the square inscribed inside the circle of radius $r = 1$ (which corresponds to B_1 in $\mathbb{R}^2$), then $V = 2$ (the area of the square) and $A = 8/\sqrt{2}$ (the perimeter of the square). Note that

$$2 < \left(\frac{1}{2}\right)\left(\frac{8}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} \approx 2.83,$$

so it is indeed the case that $V \leq \frac{1}{2}A$. Either construct or find formulas for the area and perimeter of a regular k-sided polygon inscribed in a unit circle, and show that the value of the area is less than half the value of the perimeter for every k, and further, that the ratio of area over perimeter approaches $\frac{1}{2}$ as k goes to infinity (that is, as the polygons fill up more and more of the unit circle).

6. With D, V, and A defined as in Question 4, the Isoperimetric Inequality in $\mathbb{R}^n$ states that

$$V^{(n-1)/n} \leq \left(\frac{r}{n}\right)\left(\frac{1}{\left[V\left(B_r\right)\right]^{1/n}}\right)A,$$

where $V\left(B_r\right)$ denotes the volume of the ball of radius r. For example, in $\mathbb{R}^2$, where V denotes the area of a region D and A denotes the length of its perimeter, the Isoperimetric Inequality says that

$$V \leq \left(\frac{1}{4\pi}\right)A^2,$$

with equality only in the case that D is a circle. Under the assumption that D is contained in the ball B_r of radius r centered at the origin of $\mathbb{R}^n$, show that the Isoperimetric Inequality again implies that

$$V \leq \left(\frac{r}{n}\right)A.$$