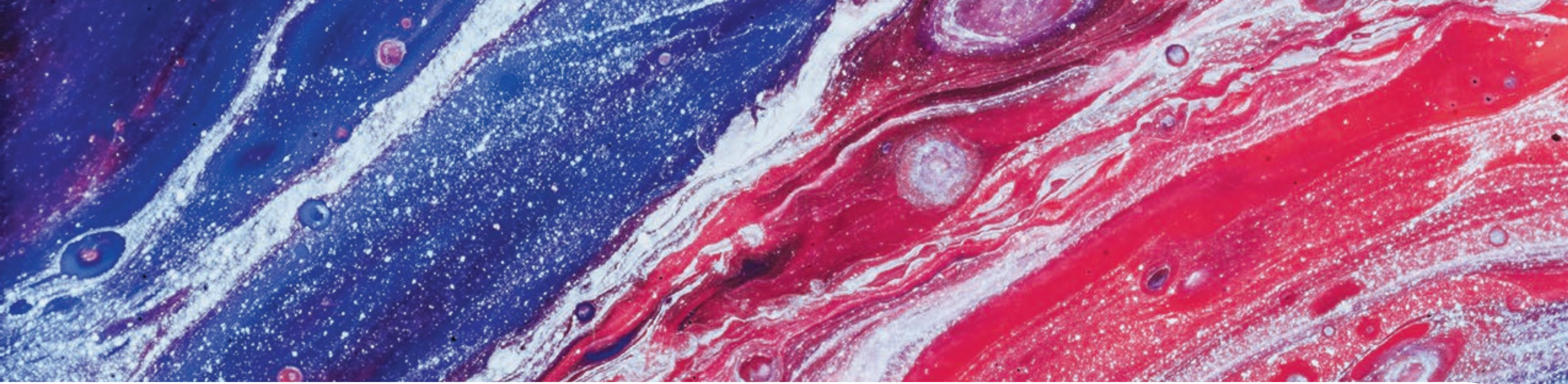

Chapter 14 Application Project: Maintaining Proper Blood Flow

In the Chapter 11 Application Project, we found the velocity profile of a fully developed laminar flow in a circular pipe. We discovered that the centerline velocity is maximum, the velocity of each streamline depends on its distance from the centerline, and that the velocity profile is parabolic.

In this project, we will use double integration to derive a formula for the average velocity of the flow and then use our findings to derive *Poiseuille's equation*. Jean Léonard Marie Poiseuille (1797–1869) was a French physicist, mathematician, and physiologist who was interested in the physics of blood circulation. He thus extensively studied the laminar flow of an incompressible fluid in a cylindrical tube, and the factors affecting such flow. Of particular interest was the fluid pressure change along the pipe, since the pressure difference dictates the pumping power required to keep the flow steady. The theory and formulas are especially important in cardiovascular applications.

We will start by determining the average velocity of the flow by considering the *volumetric flowrate*. We define the (volumetric) flowrate Q as the rate at which fluid volume passes through a particular cross-section of the tube:

$$Q = \frac{dV}{dt}.$$

1. Letting v_{av} denote the average velocity of the fluid in the pipe and A_c stand for the area of a cross-section, explain why

$$Q = v_{av} A_c,$$

that is, why the flowrate is equal to the product of the fluid's average velocity and the cross-sectional area of the pipe.

2. Suppose the radius of a person's aorta is about 1.5 cm and blood flows through it at an average velocity of $38\ \text{cm/s}$. After the aorta, blood flows through the major arteries, then successively smaller arteries and arterioles before passing through multitudes of capillaries, finally returning to the heart through venules and veins. The radius of most capillaries is about 4×10^{-4} cm, and blood passes through them at a speed of approximately $5 \times 10^{-2}\ \text{cm/s}$. Use these values to estimate the number of capillaries in the human body.

3. Use the fact that fluids are incompressible to show that

$$Q = v_{av} A_c = \iint_{A_c} v(r)\, dA,$$

where, as before, $v(r)$ is the velocity of fluid particles at distance r from the centerline.

4. Use the above equation to arrive at the formula

$$v_{av} = \frac{2}{R^2} \int_0^R v(r)\, r\, dr,$$

where R is the radius of a cross-section.

5. Substitute the formula for $v(r)$ obtained in the Chapter 11 Application Project (Question 4a) into the above integral and perform the integration to arrive at the formula

$$v_{av} = -\frac{R^2}{8\eta} \left(\frac{dP}{dx} \right).$$

6. **a.** Combine the above equation with the formula for $v(r)$ to show

$$v(r) = 2v_{av}\left(1 - \frac{r^2}{R^2}\right).$$

b. Use part a. to show that the average velocity of the flow is exactly half of its centerline velocity.

We are now ready to derive Poiseuille's equation. It shouldn't come as a surprise that pressure difference, or force, is needed to maintain flow in a horizontal pipe. This is due to the viscosity of the fluid, which creates a resistance to flow. As Poiseuille's equation will tell us, the flowrate depends on the pressure difference (more accurately called the *pressure gradient*), the viscosity of the fluid, and the dimensions of the pipe.

7. Consider a section of length L of the pipe. Denoting the fluid pressure at the initial point of the section by P_i and letting P_e stand for the pressure at the endpoint, show that the pressure change along this section satisfies

$$P_e - P_i = \frac{dP}{dx}L.$$

(**Hint:** Use the fact that $\frac{dP}{dx}$ is constant.)

8. Combine the results of Questions 1, 5, and 7 with the notation $\Delta P = P_i - P_e$ (the pressure drop along the given section of the pipe) to derive Poiseuille's equation:

$$Q = \frac{\pi R^4 (P_i - P_e)}{8\eta L} = \frac{\pi R^4 \Delta P}{8\eta L}.$$

As a last observation, notice what Poiseuille's equation tells us: The volumetric flowrate of laminar flow in a circular pipe is directly proportional to $\frac{(P_i - P_e)}{L}$ (called the pressure gradient), and inversely proportional to the viscosity of the fluid. What is more surprising, though, is the fact that Q is also directly proportional to the *fourth power* of the radius (assuming a constant pressure gradient). This explains why even a relatively small decrease in artery radius (often the result of cholesterol plaque buildup) forces the heart to work much harder to maintain the proper flowrate, with a serious side effect being *hypertension*, or high blood pressure.