

Chapter 13 Application Project: Houston, We Have Liftoff!

In the Chapter 6 Application Project, we derived the velocity function for a rocket that is applicable to its first few moments after blastoff. Our analysis assumed that gravity during the first few moments can be considered to be constant and, at the same time, is an important factor in determining the rocket's velocity. That velocity function can be expressed as

$$v(t) = v_e \ln \frac{m_0}{m(t)} - gt, \quad (1)$$

where v_e is the (positive) magnitude of the relative exhaust velocity of the expelled gas, m_0 is the initial total mass of the rocket (including fuel), $m(t)$ is the combined mass of the rocket and fuel at time t , and g is the acceleration due to gravity. After the initial blastoff phase, drag due to gravity is a less important factor in calculating change in velocity, and equation (1) is often reduced to

$$\Delta v = v_e \ln \frac{m_0}{m_f}, \quad (2)$$

where Δv represents the change in velocity over a period in which fuel is burned and the combined mass of the rocket and fuel is reduced from m_0 to m_f . Equation (2) is the classic rocket propulsion equation that was derived independently by the Russian rocket scientist Konstantin Tsiolkovsky, the American Robert H. Goddard, and the German Hermann Oberth in the first couple decades of the 20th century, and also in the early 19th century by the British mathematician William Moore.

In modern practice, booster rockets are often used to overcome drag due to gravity and atmospheric resistance in the initial blastoff phase, after which they are jettisoned. The booster rockets are attached to the sides of a central stacked column of rocket stages, each of which is also jettisoned in sequence until only the payload remains, with the goal of accelerating the payload to a final desired velocity v_f . Rocket stages (and booster rockets) are used so that the mass of the remaining rocket can be decreased as each stage is detached.

In this project, you will derive a formula for the relative masses of each stage of a two-stage rocket given the goal of minimizing total mass and achieving a given final velocity for the payload. We will assume that gravitational and atmospheric drag factors are overcome by booster rockets, and hence that only equation (2) need be considered in designing the central stack consisting of two stages and a payload.

- Let m_1 and m_2 denote the masses of, respectively, the first and second stages of the rocket when they are filled with fuel, and let P be the mass of the payload. Assume that the mass of each stage, when emptied of fuel, is the product of a structural factor s and its fuel-filled mass, where s is a positive number between 0 and 1 (s is typically less than 0.1). Then the initial mass of the stack consisting of two stages and payload is $m_1 + m_2 + P$, and the mass after the fuel of stage 1 is expended is $sm_1 + m_2 + P$. Similarly, after stage 1 is jettisoned, the initial mass of the stack consisting of stage 2 and payload is $m_2 + P$ and its mass after the fuel of stage 2 is expended is $sm_2 + P$. If we let Δv_1 denote the change in velocity of the initial stack due to the fuel of stage 1 being burned, and Δv_2 the change in velocity of the second stage and payload due to the fuel of stage 2 being burned, then the final velocity achieved by the payload is $v_f = \Delta v_1 + \Delta v_2$. Use these labels and equation (2) to express v_f as a function of m_1 , m_2 , s , v_e , and P . Name the function you construct g , and since m_1 and m_2 are the two quantities we are free to vary, we will consider g to be a function of the variables m_1 and m_2 .

- Our goal is to minimize the function $f(m_1, m_2) = m_1 + m_2 + P$ subject to the constraint $g(m_1, m_2) = v_f$, and we will use the method of Lagrange multipliers to do so. The method is difficult to apply to f and g as originally defined, but it becomes much more tractable if we make a change of variables. To that end, let

$$n_1 = \frac{m_1 + m_2 + P}{sm_1 + m_2 + P} \quad \text{and} \quad n_2 = \frac{m_2 + P}{sm_2 + P}.$$

With these definitions, show that

$$\frac{(1-s)n_1}{1-sn_1} = \frac{m_1 + m_2 + P}{m_2 + P} \quad \text{and} \quad \frac{(1-s)n_2}{1-sn_2} = \frac{m_2 + P}{P},$$

and consequently that

$$\begin{aligned} \frac{m_1 + m_2 + P}{P} &= \left(\frac{m_1 + m_2 + P}{m_2 + P} \right) \left(\frac{m_2 + P}{P} \right) \\ &= \frac{(1-s)^2 n_1 n_2}{(1-sn_1)(1-sn_2)}. \end{aligned}$$

- Note that an ordered pair $(\widetilde{m}_1, \widetilde{m}_2)$ that minimizes the function $f(m_1, m_2) = m_1 + m_2 + P$ will minimize the expression $\frac{m_1 + m_2 + P}{P}$, and hence will also simultaneously minimize the expression $\ln \frac{m_1 + m_2 + P}{P}$. The reasoning behind this is identical to the observation in Examples 3 and 4 of Section 13.8 that minimizing the square of a given distance function simultaneously minimizes the original distance function. Consequently, and because it makes the task easier, we will minimize the expression $\ln \frac{m_1 + m_2 + P}{P}$, which means we will minimize $\widehat{f}(n_1, n_2) = \ln \frac{m_1 + m_2 + P}{P} = \ln \frac{(1-s)^2 n_1 n_2}{(1-sn_1)(1-sn_2)}$

subject to the constraint

$g(n_1, n_2) = v_e \ln(n_1) + v_e \ln(n_2) = v_f$. Use the method of Lagrange multipliers to show that the minimum of $\widehat{f}$ occurs when

$$n_1 = n_2 = e^{v_f/(2v_e)}.$$

(Hint: Use properties of logarithms to rewrite $\widehat{f}$ before differentiating.)

- Show that $m_2 = \left(\frac{n_2 - 1}{1 - sn_2} \right) P$ and that

$$m_1 = \left(\frac{n_1 - 1}{1 - sn_1} \right) (m_2 + P).$$

- The Falcon 9 two-stage rocket by SpaceX is capable of lifting a payload of approximately 23 metric tons to low Earth orbit. Assuming a structural factor $s = 0.04$, exhaust velocity $v_e = 3.5$ km/s, and final velocity $v_f = 10$ km/s, determine m_1 and m_2 .