

Chapter 12 Conceptual Project: A Satellite in Motion Stays in Motion

Some orbits, such as for certain Earth satellites, can be fairly closely approximated by circles. The slight loss of accuracy in doing so is often well justified by the gain in simplicity of certain calculations. In this project, you will be guided to prove Kepler's Second and Third Laws in the case when circular orbits are assumed.

Let us now assume that a satellite is orbiting along a circular path of radius R . As we did in Section 12.4, we will start by combining Newton's Law of Universal Gravitation with his Second Law of Motion to obtain the equation of motion:

$$\mathbf{F} = m\mathbf{r}'' = m\mathbf{a} = -\frac{GMm}{r^3}\mathbf{r}.$$

1. a. Assume that the satellite's position function is

$$\mathbf{r}(t) = \langle r(t)\cos(\theta(t)), r(t)\sin(\theta(t)) \rangle,$$

where by assumption, $r(t) = R$. Recall the unit vectors $\mathbf{u}(t) = \langle \cos(\theta(t)), \sin(\theta(t)) \rangle$ and $\mathbf{u}_\perp(t) = \langle -\sin(\theta(t)), \cos(\theta(t)) \rangle$ that we used in Section 12.4, Topic 1 to describe the motion in terms of polar variables r and θ of an object with position function

$$\mathbf{r}(t) = \langle r(t)\cos(\theta(t)), r(t)\sin(\theta(t)) \rangle.$$

Explain why, in our case, $\mathbf{F} \cdot \mathbf{u}_\perp = 0$.

- b. Use the above observations and the fact that $r(t)$ is constant to prove that

$$r(t)\theta''(t) = 0.$$

2. a. Use your work above to conclude that the angular velocity

$$\omega = \frac{d\theta}{dt}$$

of the orbiting satellite is constant.

- b. Find the formula for $\frac{dA}{dt}$ in terms of ω and deduce Kepler's Second Law from the observation made in part a.

3. Use the normal component of acceleration and the curvature of the circular path (Section 12.3, Example 4) along with Newton's Second Law of Motion to show

$$\frac{GM}{R^2} = R\omega^2.$$

4. a. Use the above result to show that the period T can be expressed as

$$T = \frac{2\pi}{\omega}.$$

- b. By calculating $\frac{T^2}{R^3}$, use part a. to finish the proof of Kepler's Third Law.