Chapter 15 Conceptual Project: Showing Your Potential

Recall from Section 15.7 that if $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ so that $\nabla \times \mathbf{F} = \mathbf{0}$ (such vector fields are called curl-free) on an open, simply connected domain in space, then $\mathbf{F}$ is conservative, that is, there is a scalar potential f so that $\nabla f = \mathbf{F}$. On the other hand, it can be shown that if $\mathbf{F}$ is divergence-free, that is, if $\nabla \cdot \mathbf{F} = 0$, then there is a vector field $\mathbf{P}$ such that $\nabla \times \mathbf{P} = \mathbf{F}$ (such a vector field is called a *vector potential* for $\mathbf{F}$). In this project you will discover a way of finding a vector potential for a given divergence-free vector field $\mathbf{F}$.

1. Suppose
$$\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$
and
$$\mathbf{P}(x,y,z) = \langle P_1(x,y,z), P_2(x,y,z), P_3(x,y,z) \rangle$$
are vector fields so that $\nabla \times \mathbf{P} = \mathbf{F}$; that is, $\mathbf{P}$ is a vector potential for $\mathbf{F}$. Show that for any differentiable scalar field f, $\nabla \times (\mathbf{P} + \nabla f) = \mathbf{F}$; that is, $\mathbf{P} + \nabla f$ is another vector potential for $\mathbf{F}$. (**Hint:** See Exercise 41 of Section 15.4.)

2. If f is any scalar field such that $\dfrac{\partial f}{\partial x} = -P_1$, show that if we define $\hat{P} = P + \nabla f$, then $\hat{P}_1 = 0$.

3. Use Questions 1 and 2 to argue that if the vector field $\mathbf{F}$ has a vector potential $\mathbf{P}$, then it has one whose first component is zero. In other words, we may assume throughout our discussion that $\mathbf{P} = \langle 0, P_2, P_3 \rangle$.

In Questions 4–6, you will be guided to show that given a divergence-free vector field $\mathbf{F}$, it is possible and fairly straightforward to find a vector potential of the form described in Question 3.

4. Assume that
$$\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$
is a vector field such that $\nabla \cdot \mathbf{F} = 0$, and $\mathbf{P}$ is any vector field of the form $\mathbf{P} = \langle 0, P_2, P_3 \rangle$. Show that $\mathbf{P}$ is a vector potential for $\mathbf{F}$ if the following equalities hold.
$$\frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} = F_1 \qquad -\frac{\partial P_3}{\partial x} = F_2 \qquad \frac{\partial P_2}{\partial x} = F_3$$

5. For the vector field $\mathbf{F}$ in Question 4, define $P_2(x,y,z) = \int_{x_0}^{x} F_3(t,y,z)\,dt + C_2(y,z)$ and $P_3(x,y,z) = -\int_{x_0}^{x} F_2(t,y,z)\,dt + C_3(y,z)$, where x_0 is an arbitrary starting value and C_2 and C_3 are arbitrary functions of the variables y and z. Show that $\mathbf{P}(x,y,z) = \langle 0, P_2(x,y,z), P_3(x,y,z) \rangle$ satisfies the last two equations in Question 4.

6. Show that in Question 5, it is always possible to choose $C_2(y,z)$ and $C_3(y,z)$ to satisfy $\dfrac{\partial P_3}{\partial y} - \dfrac{\partial P_2}{\partial z} = F_1$, and conclude that $\mathbf{P}(x,y,z) = \langle 0, P_2(x,y,z), P_3(x,y,z) \rangle$ will then be a vector potential for $\mathbf{F}$. (**Hint:** Use the fact that $\nabla \cdot \mathbf{F} = 0$.)

7. Show that the vector field
$$\mathbf{F}(x,y,z) = \langle 2x^2yz, -2xy^2z, x^2y \rangle$$
is divergence-free, and follow the steps outlined in Questions 5 and 6 to find a vector potential for $\mathbf{F}$. (Answers may vary.)