# Appendices


A Fundamentals of *Mathematica* A-2

B Properties of Exponents and Logarithms, Graphs of Exponential and Logarithmic Functions A-7

C Trigonometric and Hyperbolic Functions A-8

D Complex Numbers A-13

E Proofs of Selected Theorems A-19

# Appendix A **Fundamentals of *Mathematica***

*Mathematica* is a powerful and flexible software package with a wide variety of uses. To begin with, *Mathematica* (along with similar products such as Maple, MATLAB, and Derive) can be viewed as a sort of supercalculator. It also understands the rules of algebra, has a huge number of built-in functions ranging from the trivial to the exotic, and is very good at generating high-quality graphs in one, two, and three dimensions. Beyond that, a package such as *Mathematica* is also a programming environment; it is this aspect of *Mathematica* that allows the user to extend its capabilities to suit specialized needs.

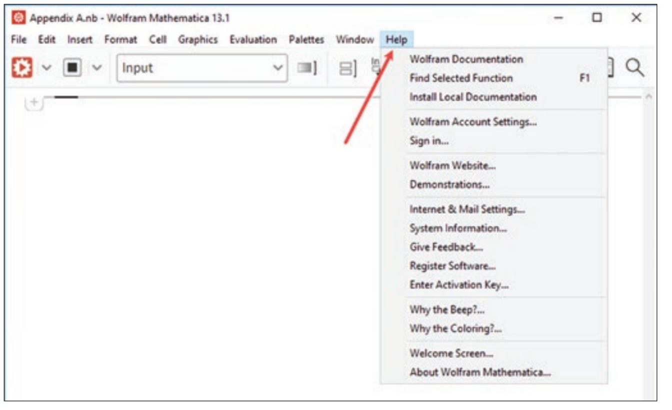



**Figure 1** Getting On-Screen Help

The optional use of *Mathematica* and similar technology in this text requires only a basic familiarity; this appendix will serve as a quick guide to the use of *Mathematica.* It should also be noted that a *complete* guide to *Mathematica* can be found within the program itself. Once it is installed and running on your computer, clicking on the "Help" button located in the top toolbar (see Figure 1) gives you access to an electronic version of a very large *Mathematica* user's manual. After clicking on "Help" a drop-down menu appears, and after clicking on "Wolfram Documentation" the full selection of "Help" categories appears. A good place to begin is with "Fast Introductions" in the Resources section of the screen. Selecting Fast Introductions will allow you to then choose "For Programmers" or "For Math Students" (either choice will open up a web page, so you must be connected to the internet to proceed). The "For Math Students" option contains a comprehensive guide and many useful examples of *Mathematica* commands, while the "For Programmers" option is appropriate for those who want to delve further into *Mathematica*'s capabilities.

At first, you will probably be making use of built-in *Mathematica* commands such as **`Plot`**, **`Fit`**, and **`Solve`** (as opposed to using your own user-defined commands). It is important to realize that *Mathematica* is case sensitive and that all built-in commands begin with a capital letter. Once a command has been typed in, you'll need to tell *Mathematica* to execute it. This can be done in one of two ways—either by pressing Shift and Enter together (known as Shift + Enter) or, if you are using an extended keyboard, by using the Enter that appears in the numeric keypad area. Pressing Enter alone will simply move the cursor to the next line and allow you to continue typing but will not execute any commands.

Each time you press Shift + Enter, *Mathematica* will execute all the commands contained in a single cell. Different *Mathematica* cells are demarcated by brackets along the right-hand edge of the work area, and you can always start a new cell by positioning the mouse cursor over a blank part of the area (you will notice that the cursor symbol becomes horizontal rather than vertical) and clicking the left mouse button once.

The remainder of this appendix contains examples of a few of the basic *Mathematica* commands used in this text, arranged roughly in the order in which they appear. For instant on-screen help on any command, type the command into *Mathematica* and then press F1. Doing so will bring up the relevant help pages and, more often than not, provide examples of how the command is used.

## Basic *Mathematica* Commands

### Defining Functions

A few rules of syntax must be observed in order to define your own functions in *Mathematica*. The first is that each variable serving as a placeholder in the definition must be followed by the underscore symbol "_" when it appears on the left side of the definition and without the underscore when it appears on the right. The second rule is that ":=" (a colon followed by an equal sign) is used in the definition, as opposed to "=" (see the on-screen *Mathematica* help for detailed explanations of these rules). Figure 2 illustrates the definition of the two functions $f(x)=x^2+5$ and $g(x,y)=3x-7y,$ followed by an evaluation of each.

```
In[1]:= f[x_] := x^2 + 5

In[2]:= g[x_, y_] := 3 x - 7 y

In[3]:= f[-2]

Out[3]= 9

In[4]:= g[5, 2]

Out[4]= 1
```

**Figure 2** Defining Functions

## Plot

The basic usage of the **`Plot`** command is **`Plot[`***f*,**`{`***x*, $x_{min}$, $x_{max}$**`}]`**, where $f$ is an expression in $x$ representing a function to be plotted and $x_{min}$ and $x_{max}$ define the endpoints of the interval on the $x$-axis over which $f$ is to be graphed. However, the **`Plot`** command also recognizes many options that modify the details of the resulting picture; these options are best explored via the on-screen help. Figure 3 illustrates the use of **`Plot`** in graphing the function $f(x)=x^3-x^2-3x+5$ over the interval $[-3,4]$.

```
In[5]:= Plot[x^3 - x^2 - 3 x + 5, {x, -3, 4}]
```

Out[5]=

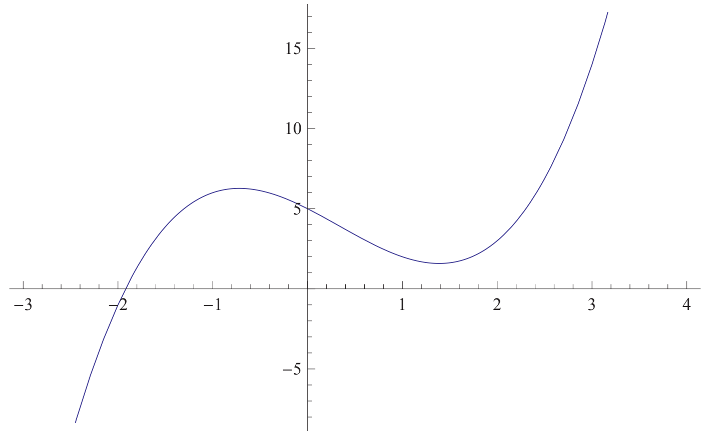



**Figure 3** Basic Use of the Plot Command

## Piecewise

The **`Piecewise`** command allows us to easily create and use functions in *Mathematica* that correspond to the piecewise-defined functions referred to in this and many other math texts. See Section 1.2 for an example of the use of the **`Piecewise`** command.

## Manipulate

The **`Manipulate`** command is a powerful tool that is useful in making dynamic models in *Mathematica*. Such models are especially useful in exploring the effect of changing the value(s) of parameter(s); see Section 1.5 for an example of such usage.

## Limit

The built-in command **`Limit`** is used to direct *Mathematica* to try to determine the limit of a function at a specified point, with the option of asking for one-sided limits from either direction. See Section 2.2 and Figure 4 for examples of the command's use.

```
In[12]:= Limit[(2 x - 1) / (x - 1), x → 1, Direction → 1]
         Limit[(2 x - 1) / (x - 1), x → 1, Direction → -1]
         Limit[(2 x - 1) / (x - 1), x → Infinity]

Out[12]= -∞

Out[13]= ∞

Out[14]= 2
```

**Figure 4a** Use of the Limit Command

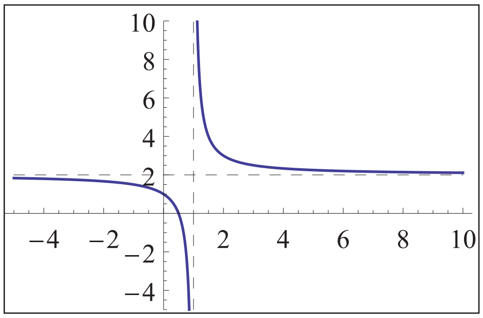



**Figure 4b** $y=\dfrac{2x-1}{x-1}$

## Differentiation (D Command)

The basic usage of the built-in differentiation command **`D`** is **`D[`***f*, *x***`]`**, where $f$ is a function of the variable $x$. Figure 5 illustrates such use in finding the derivative of a given rational function; note the optional use of the **`Together`** command (discussed later in this appendix) to express the derivative as a single fraction.

If $f$ is a function of more than one variable, the **`D`** command can be used to find partial derivatives.

```
In[17]:= f[x_] := (x^2 - 3 x + 1) / (x + 5)

In[18]:= D[f[x], x]
```

Out[18]= $\dfrac{-3+2x}{5+x}-\dfrac{1-3x+x^2}{(5+x)^2}$

```
In[19]:= Together[D[f[x], x]]
```

Out[19]= $\dfrac{-16+10x+x^2}{(5+x)^2}$

**Figure 5** Differentiation

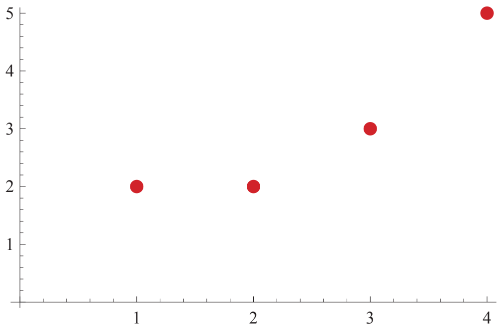
## Curve Fitting (Fit Command)

The *Mathematica* command **`Fit`** can be used to construct a function of specified form (such as linear, quadratic, exponential, etc.) to a given set of data (i.e., ordered pairs) using the least-squares method. Figure 6 illustrates the use of **`Fit`** to construct both a linear and a quadratic function that best fits the given set of four data points. Note also the use of the **`ListPlot`**, **`Plot`**, and **`Show`** commands to create graphs of the data and the two best-fitting functions. Two options are shown in the **`ListPlot`** usage, one of which (**`PlotStyle`**) specifies the color and size of the points to be plotted, and the other of which (**`AxesOrigin`**) positions the axes in a certain manner. (For an exponential fit, try out the command **`Fit[data, {1, Exp[x]}, x]`**.)

```
In[6]:= data = {{1, 2}, {2, 2}, {3, 3}, {4, 5}}
Out[6]= {{1, 2}, {2, 2}, {3, 3}, {4, 5}}

In[7]:= g1 = ListPlot[data, PlotStyle → {Red, PointSize[Large]}, AxesOrigin → {0, 0}]
```

Out[7]=




```
In[8]:= linearfit = Fit[data, {1, x}, x]
        quadraticfit = Fit[data, {1, x, x^2}, x]
Out[8]= 0.5 + 1. x
```

Out[9]= $3. - 1.5\,x + 0.5\,x^2$

```
In[10]:= g2 = Plot[{linearfit, quadraticfit}, {x, -1, 5}];
         Show[g2, g1]
```

Out[11]=

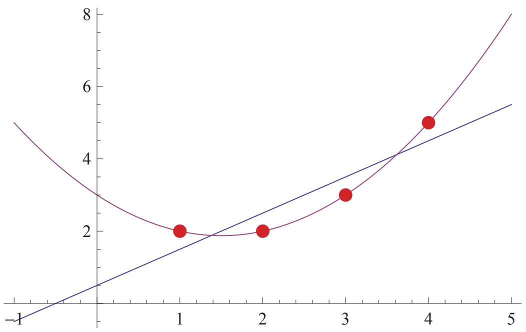



**Figure 6** Linear and Quadratic Curve Fitting

## Solve

The **`Solve`** command is very powerful, and can be used in several different ways. Its basic usage is **`Solve[`***expr***`,`** *vars***`]`**, where *expr* represents one or more equations and *vars* represents one or more variables. If more than one equation is to be solved, the collection of equations must be enclosed in a set of braces, separated by commas. Similarly, if more than one variable is to be solved for, the variables must be enclosed in a set of braces. Figure 7 shows the use of **`Solve`** to first solve one equation for one variable, and then to solve a collection of three equations for all three variables. Note how *Mathematica* expresses the solution in each case.

```
In[15]:= Solve[3 x - x * y == 9 y, y]
```

Out[15]= $\left\{\left\{y \to \frac{3\,x}{9+x}\right\}\right\}$

```
In[16]:= Solve[{3 x + 2 y - 4 z == 8, 4 x - 5 z == -3, 7 y + z == 12}, {x, y, z}]
```

Out[16]= $\left\{\left\{x \to -\frac{50}{3},\ y \to \frac{53}{15},\ z \to -\frac{191}{15}\right\}\right\}$

**Figure 7** Two Uses of the Solve Command

It is important to note that equations in *Mathematica* are expressed with two "=" symbols, as seen in Figure 7. The use of just one "=" is reserved for assigning a permanent value to something. For instance, the expression **`x=3`** assigns the value of 3 to the symbol $x$, while the expression **`x==3`** represents the equation $x = 3$ in *Mathematica*.

## NSolve

The **`NSolve`** command is used in a manner similar to **`Solve`**, but typically in situations where an exact solution is either not desired or not feasible. See Section 2.5 for an example of the use of the command in finding a numerical approximation of a solution.

## FindRoot

The **`FindRoot`** command uses numerical methods (such as Newton's method, Section 4.5) to find approximate roots of functions, and is especially useful when neither **`Solve`** nor **`NSolve`** is able to provide a satisfactory result. Its basic usage is **`FindRoot[`***f***`, {`**$x$**`, `**$x_0$**`}]`** when the goal is to find a root of the function $f$ near a given point $x_0$, but it can also be used to find a numerical solution of the equation $lhs = rhs$ near $x_0$ if used in the form **`FindRoot[`***lhs* **`==`** *rhs***`, {`**$x$**`, `**$x_0$**`}]`** (note the "double equal sign" used by *Mathematica* to denote an equation).

## FindMaximum and FindMinimum

The usage of the commands **`FindMaximum`** and **`FindMinimum`** is similar to that of **`FindRoot`**, and both also rely on numerical methods to obtain results. To approximate the location and value of a local maximum of the function $f$ near a given point $x_0$, the syntax is **`FindMaximum[`***f***`, {`**$x$**`, `**$x_0$**`}]`**; the use of **`FindMinimum`** is identical. Figure 8 illustrates the use of **`FindMinimum`** to identify the radius $r$ that minimizes the surface area of the cylinder of Example 3 in Section 4.6.

```
In[20]:= FindMinimum[2 * Pi * r^2 + 1000 / r, {r, 5}]
Out[20]= {348.734, {r → 4.30127}}
```

**Figure 8** Use of FindMinimum

## Integrate

The **`Integrate`** command can be used for both indefinite and definite integration, with the goal determined by the options used with the command. Figure 9 illustrates how *Mathematica* provides both the indefinite integral of the rational function $1/(x^2+1)$ and the definite integral of the same function over the interval $[-1.5, 1.5]$. (Note that *Mathematica* does not provide an arbitrary constant when evaluating indefinite integrals.)

```
In[21]:= Integrate[1 / (x^2 + 1), x]
Out[21]= ArcTan[x]

In[22]:= Integrate[1 / (x^2 + 1), {x, -1.5, 1.5}]
Out[22]= 1.96559
```

**Figure 9** Integration

# Other Useful Commands

## Simplify

The **`Simplify`** command is used to simplify mathematical expressions according to the usual rules of algebra. The basic syntax is **`Simplify[`***expr***`]`**, where *expr* is the expression to be simplified. Note the examples shown in Figure 10.

```
In[23]:= Simplify[x * (4 x - 2 x * y) / (6 x^2)]

         2 - y
Out[23]= -----
           3

In[24]:= Simplify[(a^2 - b^2) / (a - b)]

Out[24]= a + b
```

**Figure 10** Use of Simplify

## Expand

This command is used to multiply out factors in an expression. The syntax for the command is **`Expand[`***expr***`]`**. Figure 11 shows the use of the command in multiplying out the expression $(x-y)^5$.

```
In[25]:= Expand[(x - y)^5]

Out[25]= x^5 - 5 x^4 y + 10 x^3 y^2 - 10 x^2 y^3 + 5 x y^4 - y^5
```

**Figure 11** Use of Expand

## Factor

The **`Factor`** command is the reverse of the **`Expand`** command when applied to polynomials. Its basic usage is **`Factor[`***poly***`]`**, where *poly* is a polynomial expression to be factored.

## Together

The **`Together`** command is used primarily to express a sum (or difference) of two or more rational expressions as one with a common denominator, automatically canceling any common factors that may appear. The basic syntax for the command is **`Together[`***expr***`]`**.

# Appendix B **Properties of Exponents and Logarithms, Graphs of Exponential and Logarithmic Functions**

For ease of reference, the basic algebraic properties of exponents and logarithms and the general forms of exponential and logarithmic graphs appear below. Interestingly, the Scottish mathematician John Napier (1550–1617) introduced logarithms as an aid to computation, and their use led to the development of various types of slide rules and logarithm tables. It was only later that mathematicians made the connection between logarithmic and exponential functions, namely that they are inverses of each other (more precisely, an exponential function of a given base is the inverse function of the logarithmic function with the same base, and vice versa). This fact appears explicitly as the first property of logarithms below, with the other properties reflecting, directly or indirectly, the same fact.

## Properties of Exponents

Given real numbers $x$ and $y$ and positive real numbers $a$ and $b$, the following properties hold.

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $\left(a^x\right)^y = a^{xy}$
4. $\left(ab\right)^x = a^x b^x$

## Properties of Logarithms

Given positive real numbers $x$, $y$, $a$, and $b$, with $a \neq 1$ and $b \neq 1$, and real number $r$, the following properties hold.

1. $\log_a x = y \Leftrightarrow x = a^y$
2. $\log_a\left(a^x\right) = x$
3. $a^{\log_a x} = x$
4. $\log_a\left(xy\right) = \log_a x + \log_a y$
5. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
6. $\log_a\left(x^r\right) = r\log_a x$

Change of logarithmic base: $\log_b x = \dfrac{\log_a x}{\log_a b}$

Change of exponential base: $a^x = b^{\log_b\left(a^x\right)} = b^{x\log_b a}$

(in particular, $a^x = e^{\ln\left(a^x\right)} = e^{x\ln a}$)

## Graphs of Exponential and Logarithmic Functions

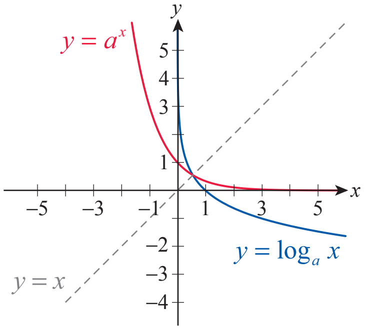



**Figure 1** Case 1: $0 < a < 1$

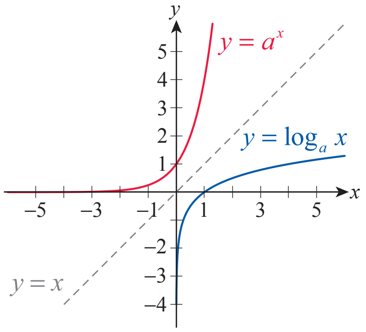



**Figure 2** Case 2: $a > 1$

# Appendix C **Trigonometric and Hyperbolic Functions**

The historical records of trigonometry date back to the second millennium BC, and we know of a number of different cultures (Egyptian, Babylonian, Indian, and Greek among them) that studied and used the properties of triangles. Our word "trigonometry" comes from an ancient Greek word meaning "triangle measuring," and the names of the individual trigonometric functions have similarly ancient roots. The study of how different cultures independently discovered the basic tenets of trigonometry, how trigonometric knowledge was further developed and disseminated, and how early civilizations used trigonometry for scientific and commercial purposes is fascinating in its own right and well worth exploring. Many excellent resources for such exploration are available online, in books, and in scholarly articles.

In contrast, the history of hyperbolic functions dates back only to the 18$^{\text{th}}$ century AD; the Italian mathematician Vincenzo Ricatti (1707–1775) and the Swiss mathematicians Johann Heinrich Lambert (1728–1777) and Leonhard Euler (1707–1783) were among the first to recognize their utility. But their development and characteristics have much in common with trigonometric functions, and they are useful today when solving differential equations and as antiderivatives of certain commonly occurring expressions.

For the purpose of quick reference, this appendix contains the basic definitions and graphs of the trigonometric and hyperbolic functions, along with frequently used identities and associated concepts.

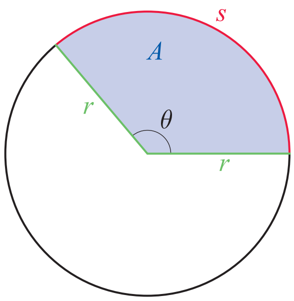
## Basic Definitions and Graphs

### Radian and Degree Measure

$$180° = \pi \text{ radians}$$

$$1° = \frac{\pi}{180} \text{ radians} \qquad 1 \text{ radian} = \frac{180°}{\pi}$$

$$x° = x\left(\frac{\pi}{180}\right) \text{ radians} \qquad x \text{ radians} = x\left(\frac{180°}{\pi}\right)$$




### Arc Length

$$s = \left(\frac{\theta}{2\pi}\right)(2\pi r) = r\theta$$

### Area of a Sector

$$A = \left(\frac{\theta}{2\pi}\right)(\pi r^2) = \frac{r^2\theta}{2}$$

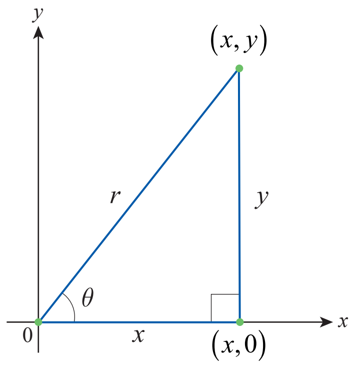
### Angular Speed

$$\omega = \frac{\theta}{t}$$

### Linear Speed

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\omega$$

### Trigonometric Functions




$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} \ (\text{for } y \neq 0)$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \ (\text{for } x \neq 0)$$

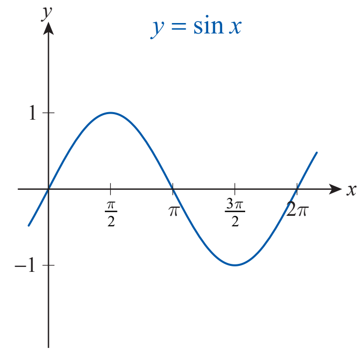
$$\tan\theta = \frac{y}{x} \ (\text{for } x \neq 0) \qquad \cot\theta = \frac{x}{y} \ (\text{for } y \neq 0)$$

## Commonly Encountered Angles

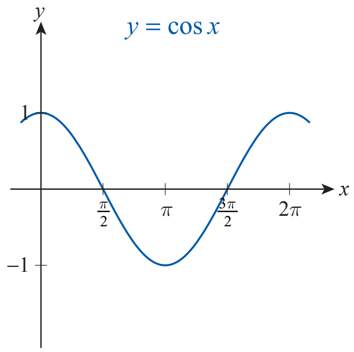
| $\theta$ | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ | $180°$ | $270°$ |
|---|---|---|---|---|---|---|---|
| **Radians** | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| **$\sin\theta$** | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ | $-1$ |
| **$\cos\theta$** | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ | $-1$ | $0$ |
| **$\tan\theta$** | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | — | $0$ | — |

## Trigonometric Graphs

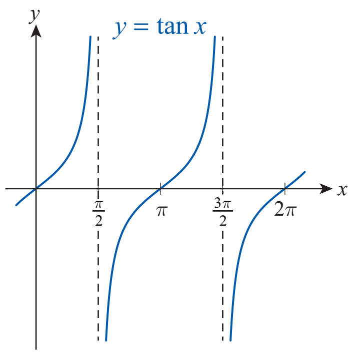
**Sine**



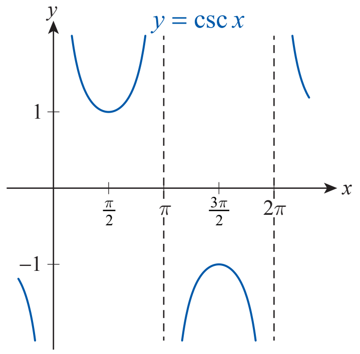

**Cosine**



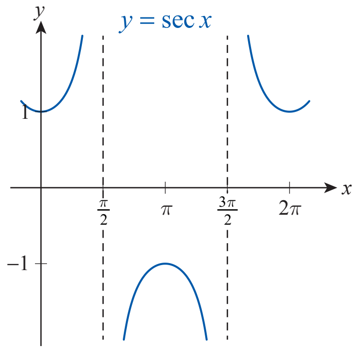

**Tangent**



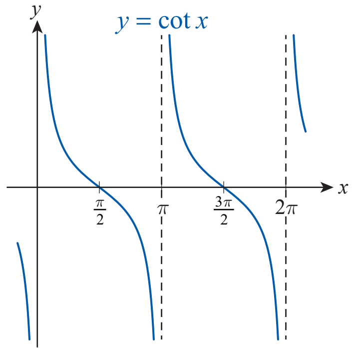

**Cosecant**




**Secant**




**Cotangent**




# Trigonometric Identities

## Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

## Cofunction Identities

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\csc x = \sec\left(\frac{\pi}{2} - x\right) \qquad \sec x = \csc\left(\frac{\pi}{2} - x\right)$$

$$\cot x = \tan\left(\frac{\pi}{2} - x\right) \qquad \tan x = \cot\left(\frac{\pi}{2} - x\right)$$

## Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

## Period Identities

$$\sin(x+2\pi) = \sin x \qquad \csc(x+2\pi) = \csc x$$
$$\cos(x+2\pi) = \cos x \qquad \sec(x+2\pi) = \sec x$$
$$\tan(x+\pi) = \tan x \qquad \cot(x+\pi) = \cot x$$

## Even/Odd Identities

$$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x \quad \sec(-x) = \sec x \quad \cot(-x) = -\cot x$$

## Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

## Sum and Difference Identities

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

## Double-Angle Identities

$$\sin 2u = 2 \sin u \cos u$$
$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

## Power-Reducing Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

## Half-Angle Identities

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

## Product-to-Sum Identities

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$
$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

## Sum-to-Product Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

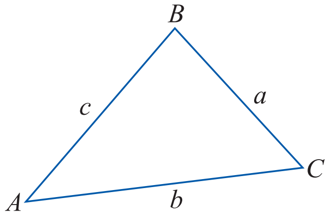
## The Laws of Sines and Cosines

**The Law of Sines**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**The Law of Cosines**

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



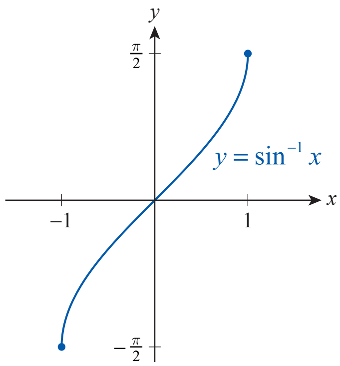

# Inverse Trigonometric Functions

## Arcsine, Arccosine, and Arctangent

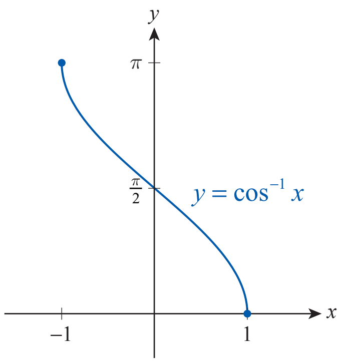
| Function | Domain | Range | Notation |
|---|---|---|---|
| **Inverse Sine** | $[-1,1]$ | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ | $\arcsin x = \sin^{-1} x = y \iff x = \sin y$ |
| **Inverse Cosine** | $[-1,1]$ | $[0,\pi]$ | $\arccos x = \cos^{-1} x = y \iff x = \cos y$ |
| **Inverse Tangent** | $(-\infty,\infty)$ | $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ | $\arctan x = \tan^{-1} x = y \iff x = \tan y$ |

## Inverse Trigonometric Graphs

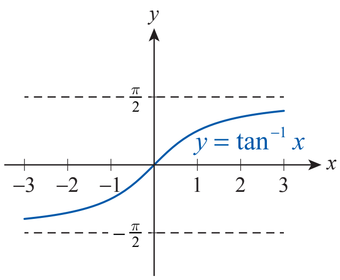
**Inverse Sine**



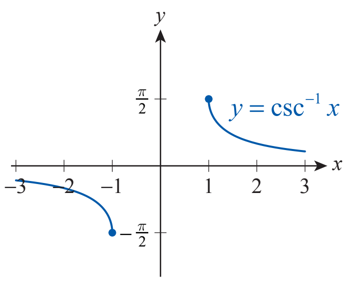

**Inverse Cosine**



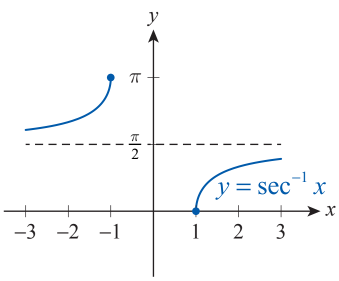

**Inverse Tangent**



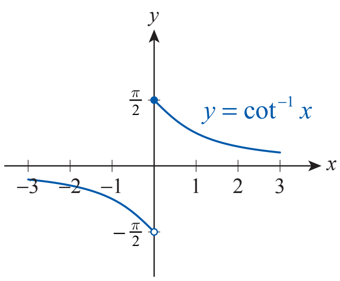

**Inverse Cosecant**




**Inverse Secant**




**Inverse Cotangent**




## Inverse Trigonometric Identities

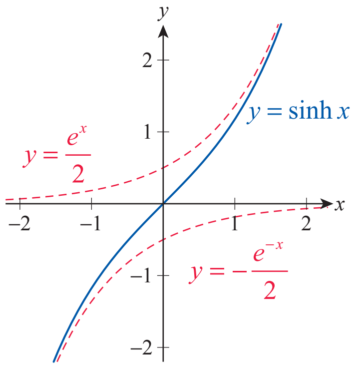
$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right) \qquad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) \qquad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), \text{ with } \cot^{-1} 0 = \frac{\pi}{2}$$

# Hyperbolic Functions

## Hyperbolic Functions and Their Graphs

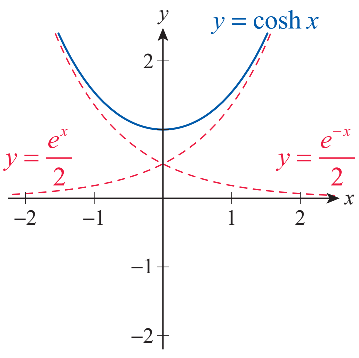
**Hyperbolic Sine**

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



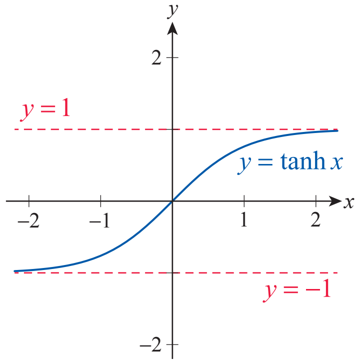

**Hyperbolic Cosine**

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



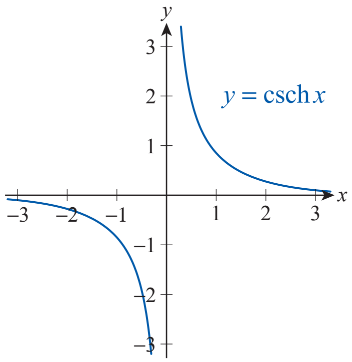

**Hyperbolic Tangent**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



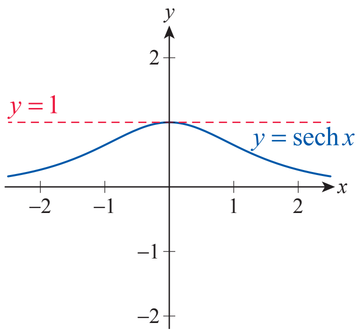

**Hyperbolic Cosecant**

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



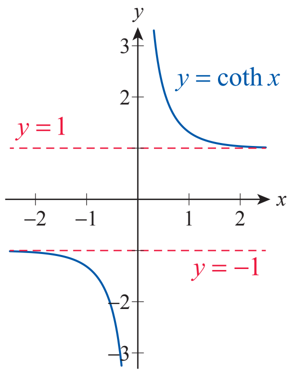

**Hyperbolic Secant**

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$




**Hyperbolic Cotangent**

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$




## Elementary Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh 2x = 2\sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x \qquad \sinh^2 x = \frac{\cosh 2x - 1}{2} \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x \qquad \cosh^2 x = \frac{\cosh 2x + 1}{2} \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

# Appendix D Complex Numbers

**A Recurring Theme**

Gauss considered the Fundamental Theorem of Algebra so important that he returned to the topic repeatedly, publishing a total of four different proofs over his lifetime—the first in 1799, two in 1816, and the fourth in 1850.

The **complex numbers**, an extension of the real numbers, consist of all numbers that can be expressed in the form $a + bi$, where $a$ and $b$ are real numbers and $i$, representing the **imaginary unit**, satisfies the equation $i^2 = -1$. Complex numbers expand the real numbers to a set that is *algebraically closed*, a concept belonging to the branch of mathematics called abstract algebra. Girolamo Cardano (1501–1576) and other Italian Renaissance mathematicians were among the first to recognize the benefits of defining what we now call complex numbers; by allowing such "imaginary" numbers as $i$, which is a solution of the equation $x^2 + 1 = 0$, mathematicians were able to devise and make sense of formulas solving polynomial equations up to degree four. Later mathematicians conjectured that every nonconstant polynomial function, even those with complex coefficients, has at least one root (a number at which the polynomial has the value of 0), assuming complex roots are allowed. Repeated application of this assertion then implies, counting multiplicities of roots, that a polynomial of degree $n$ has $n$ roots; stated another way, an $n^{\text{th}}$-degree polynomial equation has $n$ solutions (some of which may be repeated solutions). The first reasonably complete proof of this conjecture, now known as the Fundamental Theorem of Algebra, was provided by Carl Friedrich Gauss (1777–1855) in 1799 in his doctoral dissertation.

Unlike real numbers, often identified with points on a line, complex numbers are typically depicted as points in the **complex plane**, also known as the **Argand plane**, which is named after the French Swiss mathematician Jean-Robert Argand (1768–1822). The complex plane has the appearance of the Cartesian plane, with the horizontal axis referred to as the **real axis** and the vertical axis as the **imaginary axis**. A given complex number $a + bi$ is associated with the ordered pair $(a, b)$ in the plane, where $a$ represents the displacement along the real axis and $b$ the displacement along the imaginary axis (see Figure 1 for examples). In this context, $a$ is called the **real part** of $a + bi$ and $b$ the **imaginary part**. Real numbers are thus complex numbers for which the imaginary part is 0 (they can be written in the form $a + 0 \cdot i$), and **pure imaginary numbers** are complex numbers of the form $0 + bi$; the origin of the plane represents the number $0 + 0 \cdot i$ and is usually simply written as 0. Two complex numbers $a + bi$ and $c + di$ are **equal** if and only if $a = b$ and $c = d$ (that is, their real parts are equal and their imaginary parts are equal).

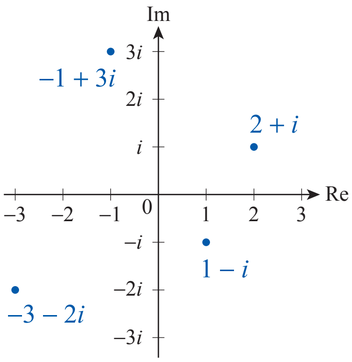



Figure 1

Sums, differences, and products of complex numbers are easily simplified and written in the form $a + bi$ by treating complex numbers as polynomial expressions in the variable $i$, remembering that $i^2 = -1$. (Keep in mind, though, that $i$ is not, in fact, a variable—this treatment is simply a convenience.) Example 1 illustrates the process.

### Example 1 Adding, Subtracting, and Multiplying Complex Numbers

Express each of the following complex expressions in the form $a + bi$.

**a.** $(4+3i)+(-5+7i)$ **b.** $(-2+3i)-(-3+3i)$

**c.** $(3+2i)(-2+3i)$ **d.** $(2-3i)^2$

**Solution**

**a.**
$$\begin{aligned}(4+3i)+(-5+7i) &= (4-5)+(3+7)i \\ &= -1+10i\end{aligned}$$

**b.**
$$\begin{aligned}(-2+3i)-(-3+3i) &= (-2+3)+(3-3)i \\ &= 1\end{aligned}$$

**c.** $(3+2i)(-2+3i) = -6+9i-4i+6i^2$

$= -6+(9-4)i-6$ Replace $i^2$ with $-1$.

$= -12+5i$

**d.** $(2-3i)^2 = (2-3i)(2-3i)$

$= 4-6i-6i+9i^2$

$= 4-12i-9$ Replace $i^2$ with $-1$.

$= -5-12i$

---

Division of complex numbers is slightly more complicated, but a quotient can also be simplified and written in the form $a + bi$ by making use of the following observation.

$$(a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$$

Given a complex number $z = a + bi$, the complex number $\overline{z} = a - bi$ is called its **complex conjugate**. We simplify a quotient of complex numbers by multiplying the numerator and denominator by the complex conjugate of the denominator, as illustrated in Example 2.

## Example 2 Simplifying Quotients of Complex Numbers

Express each of the following complex expressions in the form $a + bi$.

**a.** $\dfrac{2+3i}{3-i}$ **b.** $(4-3i)^{-1}$ **c.** $\dfrac{1}{i}$

### Solution

**a.** $\dfrac{2+3i}{3-i} = \dfrac{(2+3i)(3+i)}{(3-i)(3+i)}$ Multiply the numerator and denominator by the conjugate.

$= \dfrac{6+2i+9i+3i^2}{9+3i-3i-i^2}$

$= \dfrac{6+11i-3}{9+1}$ Replace $i^2$ with $-1$.

$= \dfrac{3+11i}{10} = \dfrac{3}{10} + \dfrac{11}{10}i$

**b.** $(4-3i)^{-1} = \dfrac{1}{4-3i}$

$= \dfrac{1(4+3i)}{(4-3i)(4+3i)}$ Multiply the numerator and denominator by the conjugate.

$= \dfrac{4+3i}{16+12i-12i-9i^2}$

$= \dfrac{4+3i}{16+9} = \dfrac{4}{25} + \dfrac{3}{25}i$

**c.** $\dfrac{1}{i} = \dfrac{1(-i)}{i(-i)} = \dfrac{-i}{-i^2} = \dfrac{-i}{1} = -i$

---

Endowed with the operations of addition and multiplication, the set of complex numbers, like the set of real numbers and the set of rational numbers, form what is known as a **field**, another concept from the realm of abstract algebra. The following table summarizes the properties possessed by a field; note that each of the three sets of numbers mentioned above possesses all the properties. Also note, by way of contrast, that the set of natural numbers, the set of integers, and the set of irrational numbers are not fields, as each set fails to possess one or more of the field properties.

### Theorem Field Properties

In the following properties, $a$, $b$, and $c$ represent arbitrary elements of a given field. The first five properties apply individually to the two operations of addition and multiplication, while the last property combines the two operations.

| Name of Property | Additive Version | Multiplicative Version |
|---|---|---|
| **Closure** | $a+b$ is an element of the field | $ab$ is an element of the field |
| **Commutative** | $a+b=b+a$ | $ab=ba$ |
| **Associative** | $a+(b+c)=(a+b)+c$ | $a(bc)=(ab)c$ |
| **Identity** | $a+0=0+a=a$ | $a\cdot 1=1\cdot a=a$ |
| **Inverse** | $a+(-a)=0$ | $a\cdot\frac{1}{a}=1$, assuming $a\neq 0$ |
| **Distributive** | $a(b+c)=ab+ac$ | |

The introduction of the imaginary unit $i$ allows us to now define the **principal square root** $\sqrt{a}$ of any real number $a$, as follows: Given a positive real number $a$, $\sqrt{a}$ denotes the positive real number whose square is $a$, and $\sqrt{-a}=i\sqrt{a}$. An application of this definition explains the restriction in one of the properties of exponents (specifically, the exponent $1/2$). Recall that if $a$ and $b$ are both positive, then

$$\sqrt{ab}=(ab)^{1/2}=a^{1/2}b^{1/2}=\sqrt{a}\sqrt{b}.$$

To see why $a$ and $b$ are required to be positive, note that

$$\sqrt{(-9)(-4)}=\sqrt{36}=6,$$

but

$$\sqrt{-9}\sqrt{-4}=\left(i\sqrt{9}\right)\left(i\sqrt{4}\right)=(3i)(2i)=6i^2=-6.$$

Complex numbers can also be expressed in *polar form*, based on the polar coordinates of a given complex number in the plane. We say the **magnitude** $|z|$ of a complex number $z=a+bi$, also known as its **modulus**, **norm**, or **absolute value**, is its distance from 0 in the complex plane—that is, the nonnegative real number

$$|z|=\sqrt{a^2+b^2}.$$

The **argument** of $z$, denoted $\arg(z)$, is the radian angle $\theta$ between the positive real axis and the line joining 0 and $z$. The quantities $|z|$ and $\arg(z)$ thus play the same roles, respectively, as the polar coordinates $r$ and $\theta$ of a point in the plane. The argument of the complex number 0 is undefined, while the argument of every other complex number is not unique (any multiple of $2\pi$ added to the argument of a given complex number describes the same number, since $2\pi$ corresponds to a complete rotation around the origin). Given these definitions, and letting $\theta = \arg(z)$, the **polar form** of $z = a + bi$ is then

$$z = r(\cos\theta + i\sin\theta), \quad \text{where} \quad r = |z|$$

(see Figure 2 for a depiction of the relationship between $a$, $b$, $r$, $|z|$, and $\theta$).

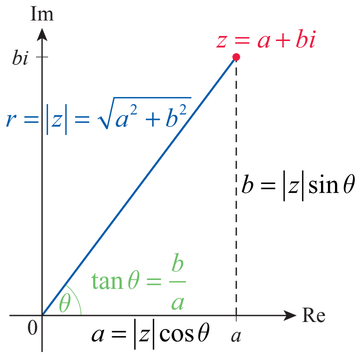



Figure 2

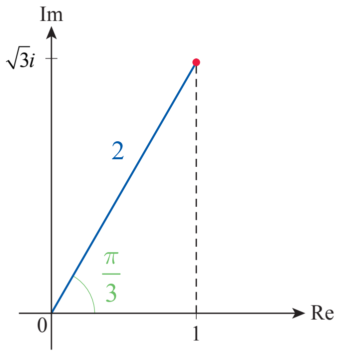



Figure 3

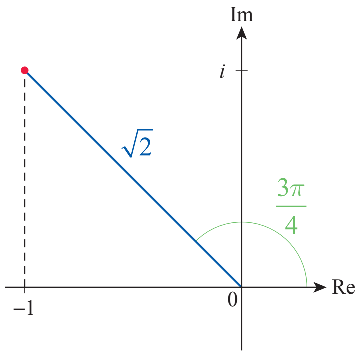



Figure 4

## Example 3 Writing Complex Numbers in Polar Form

Write each of the following complex numbers in polar form.

**a.** $1 + i\sqrt{3}$ **b.** $-1 + i$

### Solution

**a.** The magnitude of $1 + i\sqrt{3}$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$, and its argument is $\tan^{-1}\sqrt{3} = \pi/3$ (see Figure 3). Hence,

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

**b.** The magnitude of $-1 + i$ is $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$, and its argument is $3\pi/4$ (note that this complex number lies in the second quadrant of the plane, as shown in Figure 4). Hence,

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, derived in Section 10.9, allows us to express the polar form of a complex number as a complex exponential:

$$z = re^{i\theta}, \quad \text{where} \quad r = |z| \quad \text{and} \quad \theta = \arg(z).$$

With this observation, the following formulas regarding products and quotients of complex numbers are easily proved (they can also be proved by using the trigonometric sum and difference identities).

## Theorem Products and Quotients of Complex Numbers

Given the complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

the following formulas hold.

**Product Formula** $\quad z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$

**Quotient Formula** $\quad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$, assuming $z_2 \neq 0$

### Proof

Writing each complex number as a complex exponential,

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\left[\cos\left(\theta_1-\theta_2\right) + i\sin\left(\theta_2-\theta_2\right)\right].$$

The following statement regarding positive integer powers of complex numbers can be similarly proved.

### Theorem De Moivre's Theorem

Given a complex number $z = r\left(\cos\theta + i\sin\theta\right)$ and positive integer $n$,

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right).$$

### Proof

Again writing $z$ as a complex exponential,

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right).$$

De Moivre's Theorem can be used to determine roots of complex numbers. The first step is to note that if $w = re^{i\theta}$ is a nonzero complex number, and if $n$ is a positive integer, then $w$ has $n$ $n^{\text{th}}$ roots. This follows from the Fundamental Theorem of Algebra, which tells us that the equation $z^n = w$ has $n$ solutions (here, $z$ represents a complex variable). One $n^{\text{th}}$ root is easily determined: if we let

$$z_0 = r^{1/n} e^{i(\theta/n)},$$

then

$$z_0^n = \left[r^{1/n} e^{i(\theta/n)}\right]^n = re^{i\theta} = w.$$

But as we know, replacing $\theta$ with $\theta + 2k\pi$ results in an equivalent complex number for any integer $k$, leading to the following formula for the $n^{\text{th}}$ roots of $w$.

### Theorem Roots of a Complex Number

Let $w = r\left(\cos\theta + i\sin\theta\right)$ and let $n$ be a positive integer. The $n^{\text{th}}$ roots of $w$ are given by

$$z_k = r^{1/n} e^{i\left(\frac{\theta+2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

Alternatively,

$$z_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \quad k = 0, 1, \ldots, n-1.$$

The $n$ $n^{\text{th}}$ roots of a given complex number all have the same magnitude and are equally distributed around a circle in the complex plane with radius equal to that common magnitude.

## Example 4 Finding and Graphing Roots of a Complex Number

Determine the specified roots of the given complex numbers, and graph the roots and the original complex numbers in the plane.

**a.** $5^{\text{th}}$ roots of 1 **b.** $4^{\text{th}}$ roots of $-1-i\sqrt{3}$

### Solution

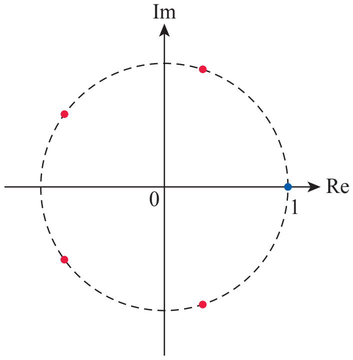



Figure 5

**a.** The easiest way to determine the $5^{\text{th}}$ roots of 1 is to write 1 as a complex exponential and then apply the above formula with $n = 5$:

$$1 = e^{i\cdot 0},$$

so the $5^{\text{th}}$ roots of 1 are

$$\left\{1, e^{i(2\pi/5)}, e^{i(4\pi/5)}, e^{i(6\pi/5)}, e^{i(8\pi/5)}\right\}.$$

Since the complex number 1 (shown as a blue point in Figure 5) has a magnitude of 1, all of the five $5^{\text{th}}$ roots (shown as red points in the figure) lie on a circle of radius 1. Note that 1 is, itself, one of the $5^{\text{th}}$ roots of 1.

**b.** The first step is to again express the complex number as a complex exponential. Note that

$$\left|-1-i\sqrt{3}\right| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

and

$$\tan\theta = \frac{-\sqrt{3}}{-1} \quad \Rightarrow \quad \theta = \frac{4\pi}{3},$$

so

$$-1-i\sqrt{3} = 2e^{i(4\pi/3)}.$$

Hence, the $4^{\text{th}}$ roots of $-1-i\sqrt{3}$ are

$$\left\{2^{1/4}e^{(i/4)[(4\pi/3)+2k\pi]}\right\}_{k=0,1,2,3} = \left\{2^{1/4}e^{i(\pi/3)}, 2^{1/4}e^{i(5\pi/6)}, 2^{1/4}e^{i(4\pi/3)}, 2^{1/4}e^{i(11\pi/6)}\right\}.$$

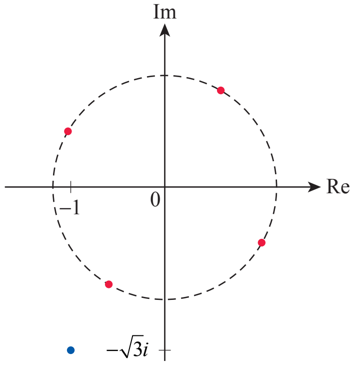



Figure 6

Figure 6 shows the original point $-1-i\sqrt{3}$ in blue and its four $4^{\text{th}}$ roots in red.

# Appendix E **Proofs of Selected Theorems**

In this appendix, we provide proofs (or in one case just a statement) of theorems used in the main body of the text. While some of the proofs here are more technical in nature than those presented elsewhere, they are worth studying in order to (1) gain additional insight into the rigorous nature of mathematical thinking and (2) develop a sense of the deeper mathematics to come in later courses.

Section 2.4

## Theorem Basic Limit Laws

Let $f$ and $g$ be two functions such that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, and let $k$ be a fixed real number. Then the following laws hold.

**Sum Law** $\lim_{x\to c}\left[f(x)+g(x)\right]=\lim_{x\to c} f(x)+\lim_{x\to c} g(x)$

**Difference Law** $\lim_{x\to c}\left[f(x)-g(x)\right]=\lim_{x\to c} f(x)-\lim_{x\to c} g(x)$

**Constant Multiple Law** $\lim_{x\to c}\left[kf(x)\right]=k\lim_{x\to c} f(x)$

**Product Law** $\lim_{x\to c}\left[f(x)g(x)\right]=\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)$

**Quotient Law** $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x)\neq 0$

## Proof

We already proved the Sum Law in Section 2.4. We proceed to prove the Product Law and Quotient Law, from which the remaining laws will follow quickly.

As in the proof of the Sum Law, let $L=\lim_{x\to c} f(x)$ and $M=\lim_{x\to c} g(x)$, and assume $\varepsilon>0$ is given. Our goal is to show there exists $\delta>0$ such that $\left|f(x)g(x)-LM\right|<\varepsilon$ for all $0<|x-c|<\delta$. One way to determine $\delta$ is to employ a strategy of adding and subtracting the same quantity, in this case $f(x)M$.

$$
\begin{aligned}
\left|f(x)g(x)-LM\right| &= \left|f(x)g(x)-f(x)M+f(x)M-LM\right| \\
&\le \left|f(x)\left[g(x)-M\right]\right|+\left|M\left[f(x)-L\right]\right| && \text{Triangle Inequality} \\
&= \left|f(x)\right|\left|g(x)-M\right|+|M|\left|f(x)-L\right|
\end{aligned}
$$

Since $L=\lim_{x\to c} f(x)$, we know there exists $\delta_1>0$ for which

$$0<|x-c|<\delta_1 \quad\Rightarrow\quad \left|f(x)-L\right|<1,$$

so $\left|f(x)\right|=\left|f(x)-L+L\right|\le\left|f(x)-L\right|+|L|<1+|L|$.

Similarly, there exists $\delta_2>0$ for which

$$0<|x-c|<\delta_2 \quad\Rightarrow\quad \left|f(x)-L\right|<\frac{\varepsilon}{2\left(1+|M|\right)}.$$

And finally, since $M=\lim_{x\to c} g(x)$, there exists $\delta_3>0$ for which

$$0<|x-c|<\delta_3 \quad\Rightarrow\quad \left|g(x)-M\right|<\frac{\varepsilon}{2\left(1+|L|\right)}.$$

So, if we let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, $0 < |x-c| < \delta$ will guarantee each of the above three outcomes, meaning

$$\begin{aligned}|f(x)g(x) - LM| &\le |f(x)||g(x) - M| + |M||f(x) - L| \\ &< (1+|L|)\frac{\varepsilon}{2(1+|L|)} + |M|\cdot\frac{\varepsilon}{2(1+|M|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

thereby proving the Product Law.

To prove the Quotient Law, we will see that it suffices to prove

$$\lim_{x\to c}\frac{1}{g(x)} = \frac{1}{M},$$

under the assumption that $M \ne 0$. In doing so, we will use the fact that $||a| - |b|| \le |a-b|$ for arbitrary real numbers $a$ and $b$. This follows from the observation that

$$|a| = |a-b+b| \le |a-b| + |b|, \qquad \text{Triangle Inequality}$$

so $|a| - |b| \le |a-b|$. Similarly, interchanging $a$ and $b$ in the same argument shows that $|b| - |a| \le |b-a| = |a-b|$, and the two facts together prove that $||a| - |b|| \le |a-b|$.

Now, given $\varepsilon > 0$, we need to show there exists $\delta > 0$ for which $0 < |x-c| < \delta$ implies

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon.$$

Since

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| = \frac{1}{|g(x)|}\cdot\frac{1}{|M|}\cdot|g(x) - M|,$$

we want to choose $\delta$ in such a manner that $|g(x) - M|$ is sufficiently small and so that $|g(x)|$ is far enough away from 0 to make

$$\frac{1}{|g(x)|}\cdot\frac{1}{|M|}$$

also sufficiently small. The fact that $M = \lim_{x\to c} g(x)$ tells us there is a $\delta_1 > 0$ for which $0 < |x-c| < \delta_1$ implies $|g(x) - M| < |M|/2$, meaning

$$0 < |x-c| < \delta_1 \;\Rightarrow\; ||g(x)| - |M|| \le |g(x) - M| < \frac{|M|}{2}, \qquad \text{Using } ||a| - |b|| \le |a-b|$$

and so

$$-\frac{|M|}{2} < |g(x)| - |M| < \frac{|M|}{2}.$$

Adding $|M|$ throughout results in the equivalent double inequality

$$\frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$$

whenever $0 < |x - c| < \delta_1$. For our present purposes, we actually only care about the fact that

$$|g(x)| > \frac{|M|}{2}, \quad \text{which means that} \quad \frac{1}{|g(x)|} < \frac{2}{|M|}.$$

There is also a $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \frac{|M|^2 \varepsilon}{2},$$

so if we let $\delta = \min\{\delta_1, \delta_2\}$, we have

$$0 < |x - c| < \delta \Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{1}{|g(x)|} \cdot \frac{1}{|M|} \cdot |g(x) - M| < \frac{2}{|M|} \cdot \frac{1}{|M|} \cdot \frac{|M|^2 \varepsilon}{2} = \varepsilon,$$

thus proving

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}.$$

We can now apply the Product Law, already proved, to obtain the Quotient Law.

$$\begin{aligned} \lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \left[ f(x) \cdot \frac{1}{g(x)} \right] \\ &= \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)} \qquad \text{Product Law} \\ &= L \cdot \frac{1}{M} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \end{aligned}$$

The Constant Multiple Law is a consequence of the Product Law, using $g(x) = k$ as one of the two functions, and the Difference Law follows from applying first the Sum Law and then the Constant Multiple Law (with $k = -1$).

Section 2.4

## Theorem Positive Integer Power Law

Let $f$ be a function for which $\lim_{x \to c} f(x)$ exists, and let $m$ be a fixed positive integer. Then

$$\lim_{x \to c} [f(x)]^m = \left[ \lim_{x \to c} f(x) \right]^m.$$

## Proof

The statement is trivially true for $m = 1$, so we prove the theorem for $m \geq 2$. We use mathematical induction to do so.

**Basis Step:** By the Product Law, we have

$$\lim_{x \to c} [f(x)]^2 = \lim_{x \to c} [f(x) f(x)] = \left[ \lim_{x \to c} f(x) \right]^2.$$

**Inductive Step:** Assume $\lim_{x \to c} [f(x)]^k = \left[ \lim_{x \to c} f(x) \right]^k$ for some $k \geq 2$. Then again applying the Product Law, we complete the proof as follows.

$$\begin{aligned}\lim_{x\to c}\left[f(x)\right]^{k+1} &= \lim_{x\to c}\left(\left[f(x)\right]^{k} f(x)\right) \\ &= \lim_{x\to c}\left[f(x)\right]^{k}\cdot\lim_{x\to c} f(x) && \text{Product Law} \\ &= \left[\lim_{x\to c} f(x)\right]^{k}\cdot\lim_{x\to c} f(x) && \text{Induction hypothesis} \\ &= \left[\lim_{x\to c} f(x)\right]^{k+1}\end{aligned}$$

Section 2.4

**Theorem** **The Squeeze Theorem**

If $g(x)\le f(x)\le h(x)$ for all $x$ in some open interval containing $c$, except possibly at $c$ itself, and if $\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L$, then $\lim_{x\to c} f(x)=L$ as well.

The statement also holds for limits at infinity, that is, for $c=-\infty$ or $c=\infty$.

**Proof**

Since $g(x)\le f(x)\le h(x)$ for all $x$ in some open interval containing $c$, there exists $\delta_1>0$ such that $0<|x-c|<\delta_1 \Rightarrow g(x)\le f(x)\le h(x)$. And by the limit definition, given $\varepsilon>0$ there exist $\delta_2>0$ and $\delta_3>0$ such that

$$0<|x-c|<\delta_2 \quad\Rightarrow\quad |g(x)-L|<\varepsilon \quad\Rightarrow\quad L-\varepsilon<g(x)<L+\varepsilon$$

and

$$0<|x-c|<\delta_3 \quad\Rightarrow\quad |h(x)-L|<\varepsilon \quad\Rightarrow\quad L-\varepsilon<h(x)<L+\varepsilon.$$

By letting $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, all three conclusions are true for $x$ within $\delta$ of $c$. That is,

$$0<|x-c|<\delta \quad\Rightarrow\quad L-\varepsilon<g(x)\le f(x)\le h(x)<L+\varepsilon \quad\Rightarrow\quad |f(x)-L|<\varepsilon.$$

Section 2.4

**Theorem** **Upper Bound Theorem**

If $f(x)\le g(x)$ for all $x$ in some open interval containing $c$, except possibly at $c$ itself, and if the limits of $f$ and $g$ both exist at $c$, then

$$\lim_{x\to c} f(x)\le\lim_{x\to c} g(x).$$

**Proof**

Let $L=\lim_{x\to c} f(x)$ and $M=\lim_{x\to c} g(x)$. Note that, by the Difference Law, the limit of $g(x)-f(x)$ at $c$ exists, and

$$\lim_{x\to c}\left[g(x)-f(x)\right]=\lim_{x\to c} g(x)-\lim_{x\to c} f(x)=M-L.$$

Suppose, in contradiction to the claim, that $L>M$. Then $L-M>0$, and if we let $\varepsilon=L-M$, there exists $\delta>0$ such that

$$\begin{aligned} 0<|x-c|<\delta \quad &\Rightarrow \quad |g(x)-f(x)-(M-L)|<\varepsilon \\ &\Rightarrow \quad -\varepsilon<g(x)-f(x)-M+L<\varepsilon \\ &\Rightarrow \quad M-L<g(x)-f(x)-M+L<L-M. \end{aligned}$$

In particular, $g(x)-f(x)-M+L<L-M$, so $g(x)-f(x)<0$ for all $x$ such that $0<|x-c|<\delta$, contradicting the fact that $f(x)\le g(x)$ for all $x$ in some open interval containing $c$. Thus, it must be the case that $L\le M$; that is, $\lim_{x\to c} f(x)\le \lim_{x\to c} g(x)$.

Section 2.5

## Theorem "Limits Pass through a Continuous Function"

Suppose $\lim_{x\to c} g(x)=a$ and $f$ is continuous at the point $a$. Then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(a).$$

In words, we say the limit operation passes inside the continuous function $f$.

## Proof

Assume $\varepsilon>0$ is given. Since $f$ is continuous at $a$, there exists $\delta_1>0$ for which

$$|x-a|<\delta_1 \quad \Rightarrow \quad |f(x)-f(a)|<\varepsilon.$$

And since $\lim_{x\to c} g(x)=a$, there exists $\delta>0$ such that

$$0<|x-c|<\delta \quad \Rightarrow \quad |g(x)-a|<\delta_1.$$

Putting these facts together, we see that

$$0<|x-c|<\delta \quad \Rightarrow \quad |g(x)-a|<\delta_1 \quad \Rightarrow \quad |f(g(x))-f(a)|<\varepsilon,$$

and hence $\lim_{x\to c} f(g(x))=f(a)$.

Section 2.5

## Theorem "The Inverse of a Continuous Function Is Continuous"

If $f$ is one-to-one and continuous on the interval $(a,b)$, then $f^{-1}$ is also a continuous function.

## Proof

We first show that $f$ is strictly monotonic on $(a,b)$, and we do so by applying the Intermediate Value Property to a number of cases, all of which are similar. If $f$ is neither strictly increasing nor strictly decreasing, then there must be points $x_1<x_2<x_3$ in $(a,b)$ for which $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. We will show that cannot happen, using a proof by contradiction.

To that end, suppose $x_1$, $x_2$, and $x_3$ *are* three points in $(a,b)$ for which $x_1<x_2<x_3$ and for which $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. Since $f$ is one-to-one, either $f(x_1)<f(x_3)$ or $f(x_1)>f(x_3)$; we will assume that $f(x_1)<f(x_3)$ and leave consideration of the other case to the reader. The assumption that $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$ again leads to two cases, one of which is that $f(x_1)<f(x_3)<f(x_2)$. Let $y$ be a value such that $f(x_3)<y<f(x_2)$.

Then by the continuity of $f$ and the Intermediate Value Property (see Section 2.5), there is a point $p$ such that $x_2 < p < x_3$ and $f(p) = y$ (in words, there is a point between $x_2$ and $x_3$ at which $f$ takes on the value $y$, since $y$ lies between the values of $f$ at $x_2$ and $x_3$). But since $f(x_1) < f(x_3)$, $y$ also satisfies $f(x_1) < y < f(x_2)$, so there is a point $q$ such that $x_1 < q < x_2$ and $f(q) = y$. But then $p \neq q$ (since $x_2$ lies strictly between them) and $f(p) = f(q)$, contradicting the fact that $f$ is one-to-one. By the same reasoning, the possibility that $f(x_2) < f(x_1) < f(x_3)$ is also ruled out, as are the two cases for which $f(x_1) > f(x_3)$. Thus, $f$ must be either strictly increasing or strictly decreasing.

To now show that $f^{-1}$ is continuous we will assume $f$ is strictly increasing—the argument that $f^{-1}$ is continuous when $f$ is strictly decreasing is similar in nature. Let $y_0$ be a point in the image of $(a,b)$ under $f$, and let $\varepsilon > 0$ be given. Since $y_0$ is in the image set, there is a (unique) point $x_0 \in (a,b)$ for which $f(x_0) = y_0$. Define

$$\varepsilon_1 = \min\{\varepsilon, x_0 - a, b - x_0\}.$$

Then, since we are assuming $f$ is increasing, the image of the interval $(x_0 - \varepsilon_1, x_0 + \varepsilon_1)$ is the interval $(f(x_0 - \varepsilon_1), f(x_0 + \varepsilon_1))$, and $y_0 \in (f(x_0 - \varepsilon_1), f(x_0 + \varepsilon_1))$. Choose $\delta > 0$ small enough so that

$$(y_0 - \delta, y_0 + \delta) \subset (f(x_0 - \varepsilon_1), f(x_0 + \varepsilon_1)).$$

Then for any $y$ such that $|y - y_0| < \delta$, $y \in (f(x_0 - \varepsilon_1), f(x_0 + \varepsilon_1))$ and hence $f^{-1}(y) \in (x_0 - \varepsilon_1, x_0 + \varepsilon_1)$. That is,

$$|y - y_0| < \delta \quad \Rightarrow \quad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon_1 \leq \varepsilon$$

and hence $f^{-1}$ is continuous at $y_0$. Since $y_0$ was arbitrary, we have shown that $f^{-1}$ is continuous on the image of $(a,b)$ under $f$.

Section 4.1

### Theorem Bolzano-Weierstrass Theorem (Statement Only)

Every bounded sequence of real numbers has a convergent subsequence.

The Bolzano-Weierstrass theorem has many uses, one of which is to help prove the Extreme Value Theorem of Chapter 4. Specifically, its use assures the existence of points in a closed and bounded interval at which a continuous function attains its extreme values.

Section 10.8

### Theorem Taylor's Theorem

If $f$ and its derivatives up through $f^{(n)}$ are all continuous on the closed interval $[a,b]$ and if $f^{(n+1)}$ exists on the open interval $(a,b)$, then there is a number $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

**Proof**

The preceding statement is actually just one variant of Taylor's Theorem, and each variant has, in turn, several slightly different methods of proof. We will use Cauchy's Mean Value Theorem (Section 4.4) to prove this variant.

We begin by defining

$$F(x)=f(x)+f'(x)(b-x)+\frac{f''(x)}{2!}(b-x)^2+\cdots+\frac{f^{(n)}(x)}{n!}(b-x)^n$$

and

$$G(x)=(b-x)^{n+1}.$$

Given these definitions, we have the following.

$$F(b)=f(b)$$

$$F(a)=f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(b-a)^n$$

$$G(b)=0$$

$$G(a)=(b-a)^{n+1}$$

Note also that

$$\begin{aligned}F'(x)&=f'(x)+\left[f''(x)(b-x)-f'(x)\right]+\left[\frac{f'''(x)}{2!}(b-x)^2-f''(x)(b-x)\right]\\&\quad+\cdots+\left[\frac{f^{(n+1)}(x)}{n!}(b-x)^n-\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}\right]\\&=\frac{f^{(n+1)}(x)}{n!}(b-x)^n\end{aligned}$$

and

$$G'(x)=-(n+1)(b-x)^n.$$

The functions $F$ and $G$ satisfy the hypotheses of Cauchy's Mean Value Theorem, and therefore there is a point $c\in(a,b)$ for which

$$\frac{F'(c)}{G'(c)}=\frac{F(b)-F(a)}{G(b)-G(a)},$$

which we will rewrite in the form

$$F(b)=F(a)+\frac{F'(c)}{G'(c)}\left[G(b)-G(a)\right].$$

Making use of the above notes, we have the following result.

$$\begin{aligned}f(b)&=F(b)\\&=F(a)+\frac{\frac{f^{(n+1)}(c)}{n!}(b-c)^n}{-(n+1)(b-c)^n}\left[-(b-a)^{n+1}\right]\\&=\underbrace{f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(b-a)^n}_{F(a)}+\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\end{aligned}$$

Section 13.3

## Theorem Clairaut's Theorem

If $f(x,y)$ and the partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are all defined on an open region containing the point $(a,b)$, and if $f_{xy}$ and $f_{yx}$ are continuous at $(a,b)$, then $f_{xy}(a,b) = f_{yx}(a,b)$.

## Proof

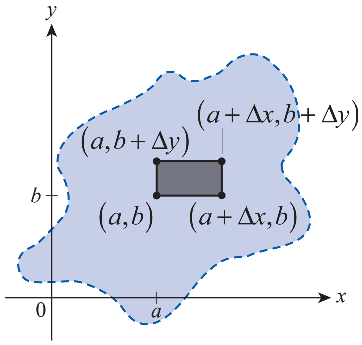



Figure 1

We begin by choosing $\Delta x \neq 0$ and $\Delta y \neq 0$ small enough so that the rectangle with vertices $(a,b)$, $(a+\Delta x, b)$, $(a+\Delta x, b+\Delta y)$, and $(a, b+\Delta y)$ is entirely contained within the open region of the hypotheses (see Figure 1), and we define

$$\Delta = \left[f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b)\right] - \left[f(a, b+\Delta y) - f(a,b)\right].$$

If we further define $g(x) = f(x, b+\Delta y) - f(x,b)$, then

$$\Delta = g(a+\Delta x) - g(a)$$

and by the Mean Value Theorem there is a point $p$ between $a$ and $a + \Delta x$ for which

$$\Delta = (\Delta x)g'(p).$$

Note that $g'(x) = f_x(x, b+\Delta y) - f_x(x,b)$, and by a second application of the Mean Value Theorem there is a point $q$ between $b$ and $b + \Delta y$ for which

$$g'(p) = (\Delta y) f_{xy}(p,q).$$

Substituting this into the expression above, we have

$$\Delta = (\Delta x)(\Delta y) f_{xy}(p,q).$$

However, if we define $h(y) = f(a+\Delta x, y) - f(a,y)$, then $\Delta$ can also be written as

$$\Delta = h(b+\Delta y) - h(b),$$

and a third application of the Mean Value Theorem tells us there is a point $s$ between $b$ and $b + \Delta y$ for which

$$\Delta = (\Delta y)h'(s).$$

Since $h'(y) = f_y(a+\Delta x, y) - f_y(a,y)$, one final application of the Mean Value Theorem implies there is a point $r$ between $a$ and $a + \Delta x$ for which

$$h'(s) = (\Delta x) f_{yx}(r,s),$$

and hence

$$\Delta = (\Delta y)(\Delta x) f_{yx}(r,s).$$

Equating the two expressions for $\Delta$, we have $f_{xy}(p,q) = f_{yx}(r,s)$, where $(p,q)$ and $(r,s)$ are both inside the rectangle with dimensions $\Delta x$ and $\Delta y$. The continuity of both $f_{xy}$ and $f_{yx}$ at $(a,b)$ means that

$$f_{xy}(p,q) = f_{xy}(a,b) + \varepsilon_1 \quad \text{and} \quad f_{yx}(r,s) = f_{yx}(a,b) + \varepsilon_2$$

where each of $\varepsilon_1$ and $\varepsilon_2$ approach 0 as $(\Delta x, \Delta y) \to (0,0)$, and so it must be the case that

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Section 13.3

## Theorem The Increment Theorem of Differentiability

If the partial derivatives of $f(x, y)$ exist throughout an open region $R$ containing the point $(a,b)$, and if $f_x$ and $f_y$ are both continuous at $(a,b)$, then $f$ is differentiable at $(a,b)$. That is, the increment $\Delta f$ at $(a,b)$ can be written as

$$\Delta f = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both $\varepsilon_1$ and $\varepsilon_2$ approach 0 as $(\Delta x, \Delta y) \to (0,0)$. Hence, if the partial derivatives are continuous throughout $R$, $f$ is differentiable on all of $R$.

Analogous statements apply to functions of three or more variables.

## Proof

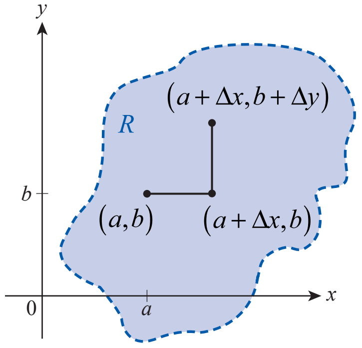



Figure 2

The increment $\Delta f$ at $(a,b)$ can be written in the form

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b),$$

and since we are interested in the form of the increment as $\Delta x$ and $\Delta y$ approach 0, we can assume they are already sufficiently small so that the line segment from $(a,b)$ to $(a+\Delta x, b)$ and the line segment from $(a+\Delta x, b)$ to $(a+\Delta x, b+\Delta y)$ both lie entirely within the open region $R$ (see Figure 2). We add and subtract $f(a+\Delta x, b)$ so that

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) + f(a+\Delta x, b) - f(a,b),$$

and we define $g(y) = f(a+\Delta x, y)$ and $h(x) = f(x,b)$. By the Mean Value Theorem, there exists a point $p$ between $b$ and $b+\Delta y$ and a point $q$ between $a$ and $a+\Delta x$ for which

$$g(b+\Delta y) - g(b) = (\Delta y)g'(p) \quad \text{and} \quad h(a+\Delta x) - h(a) = (\Delta x)h'(q).$$

Since $g'(y) = f_y(a+\Delta x, y)$ and $h'(x) = f_x(x,b)$, we now have

$$\begin{aligned}\Delta f &= f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) + f(a+\Delta x, b) - f(a,b)\\ &= g(b+\Delta y) - g(b) + h(a+\Delta x) - h(a)\\ &= (\Delta y)g'(p) + (\Delta x)h'(q)\\ &= (\Delta y)f_y(a+\Delta x, p) + (\Delta x)f_x(q,b).\end{aligned}$$

The fact that $f_x$ and $f_y$ are both continuous at $(a,b)$ means that if we define

$$\varepsilon_1 = f_x(q,b) - f_x(a,b) \quad \text{and} \quad \varepsilon_2 = f_y(a+\Delta x, p) - f_y(a,b),$$

then both $\varepsilon_1$ and $\varepsilon_2$ have limits of 0 as $(\Delta x, \Delta y) \to (0,0)$ (note that $q \to a$ as $\Delta x \to 0$ and $p \to b$ as $\Delta y \to 0$). Hence,

$$\begin{aligned}\Delta f &= (\Delta y)f_y(a+\Delta x, p) + (\Delta x)f_x(q,b)\\ &= (\Delta y)\left[f_y(a,b) + \varepsilon_2\right] + (\Delta x)\left[f_x(a,b) + \varepsilon_1\right]\\ &= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.\end{aligned}$$