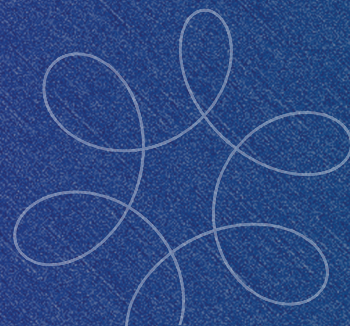


Chapter 15 Project



Recall from Section 15.7 that if \mathbf{F} is a vector field in \mathbb{R}^3 so that $\nabla \times \mathbf{F} = \mathbf{0}$ (such vector fields are called curl-free) on an open, simply connected domain in space, then \mathbf{F} is conservative, that is, there is a scalar potential f so that $\nabla f = \mathbf{F}$. On the other hand, it can be shown that if \mathbf{F} is divergence-free, that is, if $\nabla \cdot \mathbf{F} = 0$, then there is a vector field \mathbf{P} such that $\nabla \times \mathbf{P} = \mathbf{F}$ (such a vector field is called a *vector potential* for \mathbf{F}). In this project you will discover a way of finding a vector potential for a given divergence-free vector field \mathbf{F} .

1. Suppose

$$\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$$

and

$$\mathbf{P}(x, y, z) = \langle P_1(x, y, z), P_2(x, y, z), P_3(x, y, z) \rangle$$

are vector fields so that $\nabla \times \mathbf{P} = \mathbf{F}$; that is, \mathbf{P} is a vector potential for \mathbf{F} . Show that for any differentiable scalar field f , $\nabla \times (\mathbf{P} + \nabla f) = \mathbf{F}$; that is, $\mathbf{P} + \nabla f$ is another vector potential for \mathbf{F} .

(Hint: See Exercise 41 of Section 15.4.)

2. If f is any scalar field such that $\frac{\partial f}{\partial x} = -P_1$, show that if we define $\hat{\mathbf{P}} = \mathbf{P} + \nabla f$, then $\hat{P}_1 = 0$.

3. Use Questions 1 and 2 to argue that if the vector field \mathbf{F} has a vector potential \mathbf{P} , then it has one whose first component is zero. In other words, we may assume throughout our discussion that $\mathbf{P} = \langle 0, P_2, P_3 \rangle$.

In Questions 4–6, you will be guided to show that given a divergence-free vector field \mathbf{F} , it is possible and fairly straightforward to find a vector potential of the form described in Question 3.

4. Assume that

$$\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$$

is a vector field such that $\nabla \cdot \mathbf{F} = 0$, and \mathbf{P} is any vector field of the form $\mathbf{P} = \langle 0, P_2, P_3 \rangle$. Show that \mathbf{P} is a vector potential for \mathbf{F} if the following equalities hold:

$$\frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} = F_1 \quad -\frac{\partial P_3}{\partial x} = F_2 \quad \frac{\partial P_2}{\partial x} = F_3$$

5. For the vector field \mathbf{F} in Question 4, define

$$P_2(x, y, z) = \int_{x_0}^x F_3(t, y, z) dt + C_2(y, z) \text{ and}$$

$$P_3(x, y, z) = -\int_{x_0}^x F_2(t, y, z) dt + C_3(y, z),$$

where x_0 is an arbitrary starting value and C_2 and C_3 are arbitrary functions of the variables y and z . Show that $\mathbf{P}(x, y, z) = \langle 0, P_2(x, y, z), P_3(x, y, z) \rangle$ satisfies the last two equations in Question 4.

6. Show that in Question 5, it is always possible to choose $C_2(y, z)$ and $C_3(y, z)$ to satisfy $\frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} = F_1$, and conclude that

$\mathbf{P}(x, y, z) = \langle 0, P_2(x, y, z), P_3(x, y, z) \rangle$ will then be a vector potential for \mathbf{F} . (Hint: Use the fact that $\nabla \cdot \mathbf{F} = 0$.)

7. Show that the vector field

$$\mathbf{F}(x, y, z) = \langle 2x^2yz, -2xy^2z, x^2y \rangle$$

is divergence-free, and follow the steps outlined in Questions 5 and 6 to find a vector potential for \mathbf{F} . (Answers may vary.)