Chapter 14 Project

In this project you will be able to take advantage of useful coordinate transformations to evaluate multiple integrals on ellipses and ellipsoids that would be much more challenging in the Cartesian coordinate system.

1. Find the Jacobian of the coordinate transformation $T(r,\theta)$ defined by $x = ar\cos\theta$ and $y = br\sin\theta$, where a, $b > 0$.

2. Use double integration along with the coordinate transformation in Question 1 to arrive at the formula for the area A of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Find the Jacobian of the transformation to "ellipsoidal coordinates" $T_e(\rho,\theta,\phi)$ defined by $x = a\rho\sin\phi\cos\theta$, $y = b\rho\sin\phi\sin\theta$, and $z = c\rho\cos\phi$, where a, b, $c > 0$.

4. Use double integration along with the coordinate transformation of Question 3 to arrive at the formula for the volume V of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

5. Use ellipsoidal coordinates to find the center of mass of the upper ellipsoid $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, assuming constant density.

6. Find the mass and the center of mass of the semiellipsoid of Question 5 in the case that the density at any point is proportional to the distance from the xy-plane.

7. Use a computer algebra system and ellipsoidal coordinates to find the second moments and radii of gyration for the solid of Question 5. Express the second moments in terms of the mass m of the semiellipsoid.

8. Use a computer algebra system and ellipsoidal coordinates to find the second moments and radii of gyration for the solid of Question 6. As in the previous problem, express the second moments in terms of the mass m of the semiellipsoid.