Chapter 13 Project

In this project you will use your experience with partial derivatives and differentials to learn how to solve an important class of differential equations, called **exact equations**. Ordinary differential equations of this type are noted for their widespread applications in physics and engineering. (See Section 8.1 for the definitions of differential equation and solution. Other than the basic definitions, this project does not directly rely on, and can be considered independently of Chapter 8.)

1. Suppose that the first-order partial derivatives of the function $z = f(x, y)$ are both continuous on a region R. If c is a constant and $y = y(x)$ is defined implicitly by the equation $f(x, y) = c$, show that y solves the differential equation

$$f_y(x, y) \cdot y' = -f_x(x, y).$$

2. Now consider a differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

and assume that there is a two-variable function $f(x, y)$ such that

$$\frac{\partial f(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = N(x, y)$$

(such a differential equation is called *exact*, while $f(x, y)$ is called a *potential function*). Use your answer to Question 1 to show that the set of level curves $f(x, y) = C$, $C \in \mathbb{R}$ form a family of solutions of the differential equation (1).

3. Suppose that $M(x, y)$ and $N(x, y)$, as well as their first-order partial derivatives, are continuous on an open region R. Show that a necessary condition for equation (1) to be exact is the following equality.

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

(**Note:** If we require a bit more of R, the above condition is also sufficient for exactness, a statement we will not rigorously prove here, but the construction of a potential function under the stated conditions is outlined in Questions 5 and 6.)

4. Use Question 3 to determine which of the following equations is exact.

a. $\left(2x + ye^{xy}\right)dx + \left(xe^{xy} - 1\right)dy = 0$

b. $\left(3x^2 y - \frac{1}{\sqrt{x}}\right)dx + \left(x^3 - \sqrt{x}\right)dy = 0$

5. Explain why the potential function f of an exact equation must satisfy

$$f(x, y) = \int M(x, y)\,dx + g(y),$$

where g is some function of the variable y.

6. Show that if the equation (1) is exact, then the equality

$$N(x, y) = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y)$$

must hold.

7. Use Questions 5 and 6 to solve the equation $2\left(x - y^2\right)dx + y(9y - 4x)\,dy = 0$ by determining its potential function $f(x, y)$ and identifying the family of solutions as $f(x, y) = C$. (**Hint:** After identifying $M(x, y)$ and $N(x, y)$, use Question 5 to obtain a tentative formula for $f(x, y)$, then use Question 6 to determine the unknown function $g(y)$.)

8. Verify that the equation $(2 + x)y\,dx + 2x\,dy = 0$ becomes exact after multiplying by the integrating factor $I(x, y) = xye^x$. Solve the resulting equation.