# Chapter 1 Project

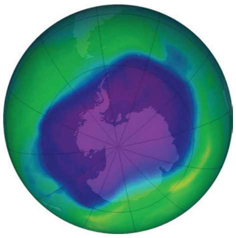


Photo Courtesy of NASA

As time goes on, there is increasing awareness, controversy, and legislation regarding the ozone layer and other environmental issues. The hole in the ozone layer over the South Pole disappears and reappears in a cyclical manner annually. Suppose that over a particular stretch of time the hole is assumed to be circular with a radius growing at a constant rate of 2.6 kilometers per hour.

**1.** Assuming that $t$ is measured in hours, that $t = 0$ corresponds to the start of the annual growth of the hole, and that the radius of the hole is initially 0, write the radius as a function of time, $t$. Denote this function by $r(t)$.

**2.** Use function composition to write the area of the hole as a function of time, $t$. Denote this function by $A(t)$. Sketch the graph of $A(t)$ and label the axes appropriately.

**3.** After finding $A(1)$, the area of the ozone hole at the end of the first hour, determine the time necessary for this area to double. How much additional time does it take to reach three times the initial area?

**4.** Are the two time intervals you found in Question 3 equal? If not, which one is greater? Explain your finding. (Use a comparison of some basic functions discussed in Section 1.2 in your explanation.)

**5.** What are the radius and area after 3 hours? After 5.5 hours?

**6.** What is the average rate of change of the area from 3 hours to 5.5 hours?

**7.** What is the average rate of change of the area from 5.5 hours to 8 hours?

**8.** Is the average rate of change of the area increasing or decreasing as time passes?

**9.** What flaws do you see with this model? Can you think of a better approach to modeling the growt of the ozone hole?

# Chapter 2 Project

Some years ago, it was common for long-distance phone companies to charge their customers in one-minute increments. In other words, the company charges a flat fee for the first minute of a call and another fee for each additional minute or any fraction thereof (see Exercise 82 in Section 2.5). In this project, we will explore in detail a function that gives the cost of a telephone call under the above conditions.

1. Suppose a long-distance call costs 75 cents for the first minute plus 50 cents for each additional minute or any fraction thereof. In a coordinate system where the horizontal axis represents time $t$ and the vertical axis price $p$, draw the graph of the function $p = C(t)$ that gives the cost (in dollars) of a telephone call lasting $t$ minutes, $0 < t \leq 5$.
2. Does $\lim_{t \to 1.5} C(t)$ exist? If so, find its value.
3. Does $\lim_{t \to 3} C(t)$ exist? Explain.
4. Write a short paragraph on the continuity of this function. Classify all discontinuities; mention one-sided limits and left or right continuity where applicable.
5. In layman's terms, interpret $\lim_{t \to 2.5} C(t)$.
6. In layman's terms, interpret $\lim_{t \to 3^-} C(t)$.
7. In layman's terms, interpret $\lim_{t \to 3^+} C(t)$.
8. If possible, find $C'(3.5)$.
9. If possible, find $C'(4)$.
10. Find and graph another real-life function whose behavior is similar to that of $C(t)$. Label the axes appropriately and provide a brief description of your function.

# Chapter 3 Project

The following table shows the atmospheric pressure $p$ at the altitude of $k$ feet above sea level (pressure is measured in mm Hg; note that this unit of pressure is approximately the pressure generated by a column of mercury 1 millimeter high).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ***k*** **(ft)** | 0 | 1000 | 2000 | 3000 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 | 10,000 |
| ***p*** **(mm Hg)** | 760 | 733 | 707 | 681 | 656 | 632 | 609 | 586 | 564 | 543 | 523 |

**1.** Find the average rate of change of air pressure from sea level to 2000 feet of altitude.

**2.** Find the average rate of change of air pressure between the altitudes of 4000 and 10,000 feet.

**3.** Use a *symmetric difference quotient*

$$\frac{p(c+h)-p(c-h)}{2h}$$

to estimate the instantaneous rate of change of air pressure at 7000 ft by choosing $h = 1000$ ft.

**4.** Tell whether you expect the answer to Question 2 or 3 to better approximate the instantaneous rate of change of air pressure at altitude 7000 ft. Explain. (**Hint:** Plotting the data on paper may help.)

**5.*** Explain why you expect the symmetric difference quotient $\dfrac{f(c+h)-f(c-h)}{2h}$ in general to be a better approximation of the instantaneous rate of change of $f$ at $x = c$ than the "regular" difference quotient $\dfrac{f(c+h)-f(c)}{h}$.

**6.** Use a graphing calculator or computer algebra system to find an exponential regression curve to the given data and plot the curve along with the data on the same screen.

**7.** Use the exponential function you found in Question 6 to estimate the instantaneous rate of change of air pressure at 7000 ft, and compare with your estimate given in Question 3.

**8.** Is the instantaneous rate of change increasing or decreasing with altitude? Explain.

# Chapter 4 Project

Consider a function $f(x)$ that is at least twice differentiable. In this project, you will show that the second derivative of $f(x)$ at $x = c$ can be found as the limit of so-called **second-order differences**:

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

**1.** Instead of working with a secant line through the points $(c, f(c))$ and $(c+h, f(c+h))$ like we did when approximating the first derivative, suppose that

$$y = a_1 x^2 + a_2 x + a_3$$

is the parabola through the following three points on the graph of $f$: $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$. Do you expect to always be able to find coefficients $a_1, a_2, a_3 \in \mathbb{R}$ such that the resulting parabola satisfies the desired conditions? Why or why not? Why would you expect $2a_1$ to be "close" to $f''(c)$ if $h$ is "small"? What will happen to $2a_1$ as $h \to 0$? Write a short paragraph answering the above questions.

**2.** By substituting the points $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$ into $y = a_1 x^2 + a_2 x + a_3$, obtain a system of linear equations in unknowns $a_1$, $a_2$, and $a_3$. Solve the system for the unknown $a_1$.

**3.** Use Questions 1 and 2 to argue that $f''(c)$ is the limit of the second-order differences:

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

**4.** Use l'Hôpital's Rule to verify the result you found in Question 3.

# Chapter 5 Project

The topic of this project is the so-called *sine integral function*, which is important for its applications, most notably in electrical engineering and signal processing.

1. Consider the following piecewise defined function:

$$f(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases}$$

   Prove that for any $x \geq 0$, $f(t)$ is integrable on $[0, x]$.

2. The **sine integral function** is defined as follows:

$$\mathrm{Si}(x) = \int_0^x f(t)\,dt, \text{ for } x \geq 0$$

   Prove that $\mathrm{Si}(x)$ is continuous.

3. Find the derivative $\dfrac{d}{dx}\mathrm{Si}(x)$.

4. Without graphing first, write a short paragraph on why you would expect the graph of $\mathrm{Si}(x)$ to be oscillating. Explain why its amplitude is expected to decrease as $x \to \infty$.

5. Find the $x$-values where the relative maxima and minima of $\mathrm{Si}(x)$ occur.

6. Extend the definition of $\mathrm{Si}(x)$ to negative $x$-values and prove that for any $a > 0$, $\int_{-a}^{a} \mathrm{Si}(x)\,dx = 0$.

7. Use a computer algebra system to plot the graph of $\mathrm{Si}(x)$ on the interval $[-8\pi, 8\pi]$.

8. Use a computer algebra system to approximate the range of $y = \mathrm{Si}(x)$ to four decimal places.

# Chapter 6 Project

In this project, we will expand upon our explorations from Exercises 48 through 55 of Section 6.5 (also see the discussion preceding those exercises). In particular, we will determine an equation satisfied by the velocity of a projectile launched with initial velocity $v_0$, taking into consideration that acceleration caused by gravity decreases with altitude. (This is important when objects are launched to great altitudes.) We will then use our equation to find the maximum height attained by the projectile. This will lead us to the value of the so-called *escape velocity*, the velocity needed for an object to be able to overcome the Earth's gravitational field without further propulsion. (In turn, since gravity is conservative, this is the same velocity an object would achieve if pulled in by gravity from an "infinite distance.") We are ignoring all retarding forces (such as air resistance or friction) in this discussion.

**1.** Recall from Exercise 48 of Section 6.5 that $g$, the acceleration caused by gravity on a free-falling mass near the Earth's surface, is approximately

$$g = \frac{MG}{R^2},$$

where $M$ and $R$ are the mass and radius of the Earth, respectively, and $G$ is the universal gravitational constant. However, a launched projectile's acceleration caused by gravity is negative (if we are assuming the positive direction is upwards) and actually depends on its height $h$ above the Earth's surface. In particular, use Newton's Law of Gravitation to show that this dependence is given by the equation

$$a(h) = \frac{-gR^2}{(R+h)^2},$$

where $a(0) = -g$, as we would expect. (Actually, $a(h) \approx -g$ when $h$ is negligible compared to the Earth's radius.)

**2.** Show that if $v = v(h)$ denotes the velocity of the projectile, then

$$\frac{d}{dh}(v^2) = 2\frac{dv}{dt}.$$

(**Hint:** Use the Chain Rule.)

**3.** Use the above results to show that

$$\frac{d(v^2)}{dh} = \frac{-2gR^2}{(R+h)^2}.$$

**4.** Integrating both sides of the previous equation with respect to $h$, show that $v = v(h)$ satisfies the equation

$$v^2 = v_0^2 - 2gR\left(1 - \frac{R}{R+h}\right).$$

(**Hint:** After integrating, use the fact that $v(0) = v_0$.)

**5.** Use the equation found in Question 4 to find the maximum height attained by the projectile. (**Hint:** Use the fact that $v = 0$ when the projectile reaches its maximum height.)

**6.** Find a formula for the escape velocity $v_e$ of the projectile; then use the data found in the exercises of Section 6.5 (Exercises 48–55 and the preceding discussion) to express your answer in kilometers per second. (**Hint:** Use the fact that if $v_0 = v_e$, the projectile will "travel to infinity.")

**7.** Find the escape velocity of the projectile if it is launched on the Moon. (**Hint:** For Moon data, see Exercise 53 of Section 6.5.)

# Chapter 7 Project

In this project, we will derive a famous infinite product named after its discoverer, the English mathematician John Wallis (1616–1703). Wallis introduced the symbol $\infty$ for infinity, and in turn he used $1/\infty$ to denote an *infinitesimal* quantity. He contributed to the development of *infinitesimal calculus* (it wasn't until the 19th century that infinitesimals were replaced by limits in the works of Bolzano, Cauchy, and Weierstrass).

**1.** For a nonnegative integer $n$, let

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx.$$

Find $I_0$, $I_1$, $I_2$, and $I_3$.

**2.** Show that if $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(**Hint:** See Exercise 81 of Section 7.1.)

**3.** Use Questions 1 and 2 to find $I_4$, $I_5$, $I_6$, and $I_7$.

**4.** Show that in general,

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

while

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \cdots \cdot \frac{2}{3}.$$

(**Hint:** Observe a pattern or use induction.)

**5.** Use Question 4 to show that

$$\frac{I_{2n}}{I_{2n+1}} = \frac{3^2 5^2 \cdot \cdots \cdot (2n-1)^2 (2n+1)}{2^2 4^2 \cdot \cdots \cdot (2n)^2} \cdot \frac{\pi}{2}$$

holds for all $n$.

**6.** Show that

$$\frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n}.$$

**7.** Prove the inequalities

$$I_{2n-1} \geq I_{2n} \geq I_{2n+1}.$$

(**Hint:** Use the definition of $I_n$ from Question 1 and compare the integrands.)

**8.** Use Questions 6 and 7 to show that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n},$$

and use this observation to prove that

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

**9.** Use your answers to the previous questions to derive **Wallis' product**:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^2 4^2 \cdot \cdots \cdot (2n)^2}{3^2 5^2 \cdot \cdots \cdot (2n-1)^2 (2n+1)}$$

# Chapter 8 Project

Recall from Section 3.7 our discussion of a chemical reaction where reactants $A$ and $B$ produce a new product substance $C$, a process represented by

$$A + B \to C.$$

In this project, we will derive and use a differential equation that describes such a process.

1. Suppose that in the above reaction for each gram of reactant $A$, $b$ grams of $B$ are used to form $C$. If we start with initial amounts $A_0$ and $B_0$, respectively, and $X(t)$ denotes in grams the amount of substance $C$ already formed at time $t$, find the amount remaining of reactants $A$ and $B$ at any time during the process.

2. Given that the rate of formation of substance $C$ at any time is proportional to the product of the remaining amounts of reactants $A$ and $B$, respectively, find a differential equation in terms of $X(t)$ that describes the process.

   (As in Question 1, let $A_0$ and $B_0$ stand for the initial amounts.)

3. Suppose a product substance $C$ is being formed from reactant substances $A$ and $B$ and that for each gram of substance $A$, 3 grams of $B$ are used to form $C$. As in Question 1, let $X(t)$ denote the amount of $C$ formed at time $t$, and assume that the initial amounts of reactants $A$ and $B$ are $A_0 = 60$ grams and $B_0 = 40$ grams, respectively. Find the initial value problem describing this reaction. (**Hint:** Use your answer to Question 2.)

4. If 20 grams of the product compound forms during the first 5 minutes, use the model you obtained in Question 3 to predict how much of the product compound $C$ is present 10 minutes into the process.

5. Use your model from Question 3 to predict what happens as $t \to \infty$. Interpret your answer.

# Chapter 9 Project

In this project, you will be introduced to a class of parametric curves called *Bézier curves*. They are important for their applications in engineering, computer graphics, and animation. This class of curves is named after Pierre Bézier (1910–1999), a design engineer for the French automaker Renault, who first demonstrated these curves' use in designing automobile bodies in the 1960s. The design advantage of Bézier curves lies in the fact that they can easily be manipulated by moving around their so-called *control points*. In addition, it is easy to smoothly join together several Bézier curves for more complicated shapes.

1. The linear Bézier curve $B_{0,1}(t)$ from $P_0(a_0,b_0)$ to $P_1(a_1,b_1)$ is simply the line segment connecting the two points (note that $P_0$ and $P_1$ are the only control points in this case). Verify that this curve can be parametrized as

$$B_{0,1}(t)=(1-t)P_0+tP_1,\ t\in[0,1],$$

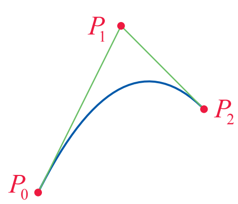
and find $x(t)$ and $y(t)$ corresponding to this parametrization. (In this and subsequent questions, control points will be labeled $P_i(a_i,b_i)$, $0\le i\le 3$.)

2. The Bézier curve $B_{0,1,2}(t)$ with control points $P_0$, $P_1$, and $P_2$ is a quadratic curve joining the points $P_0$ and $P_2$ in such a way that both line segments $\overline{P_0P_1}$ and $\overline{P_1P_2}$ are tangent to $B_{0,1,2}(t)$. Intuitively speaking, this means that the curve "starts out at $P_0$ in the direction of $P_1$," and "arrives at $P_2$ from the direction of $P_1$" (see figure).

   Find $x(t)$ and $y(t)$ corresponding to the parametrization

$$B_{0,1,2}(t)=(1-t)B_{0,1}(t)+tB_{1,2}(t),\ \ t\in[0,1]$$

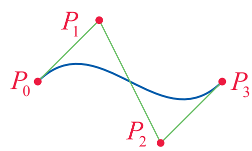
   and verify that $B_{0,1,2}(t)$ satisfies the conditions stated above.




A Quadratic Bézier Curve

3. The cubic Bézier curve $B_{0,1,2,3}(t)$ with control points $P_0$, $P_1$, $P_2$, and $P_3$ joins $P_0$ and $P_3$ so that the line segments $\overline{P_0P_1}$ and $\overline{P_2P_3}$ are tangent to $B_{0,1,2,3}(t)$ at $P_0$ and $P_3$, respectively (see figure). Verify that the following curve satisfies these conditions:

$$x(t)=a_0(1-t)^3+3a_1(1-t)^2t+3a_2(1-t)t^2+a_3t^3$$
$$y(t)=b_0(1-t)^3+3b_1(1-t)^2t+3b_2(1-t)t^2+b_3t^3,\ t\in[0,1]$$




A Cubic Bézier Curve

4. Show that the parametrization in Question 3 corresponds to

$$B_{0,1,2,3}(t)=(1-t)B_{0,1,2}(t)+tB_{1,2,3}(t).$$

5. Use Question 3 to verify that the Bézier curve with control points $P_0(1,3)$, $P_1(3,7)$, $P_2(6,9)$, and $P_3(8,6)$ has the following parametrization:

$$x(t)=-2t^3+3t^2+6t+1$$
$$y(t)=-3t^3-6t^2+12t+3$$

6. Find the slope of the curve in Question 5 at **a.** $t=0$, **b.** $t=\frac{1}{2}$, and **c.** $t=1$.

7. Use a computer algebra system to graph the Bézier curve of Question 5 along with its control points. If your CAS has animation capabilities, explore what happens if you move around the control points in the plane.

# Chapter 10 Project

In this project, we are going to expand on our earlier work with the harmonic series. In the process, we will meet a famous constant called *Euler's constant*, also known as the *Euler-Mascheroni constant*. (This number is not to be confused with $e \approx 2.71828$, the natural base, which is also known as Euler's number.)

**1.** As in Example 6 of Section 10.2, we let $s_n$ stand for the $n^{\text{th}}$ partial sum of the harmonic series; that is,

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(The partial sum $s_n$ is also called the $n^{\text{th}}$ *harmonic number*.) For each $n \geq 1$, we define

$$d_n = s_n - \ln n.$$

Prove that $d_n > 0$ for any positive integer $n$. (**Hint:** Refer to the illustration provided for Exercise 65 of Section 10.2, and start by comparing $s_n$ with $\int_1^{n+1} (1/x)\,dx$.)

**2.** Prove that $\{d_n\}$ is a decreasing sequence. (**Hint:** Referring again to the figure from Exercise 65 of Section 10.2, fix an $n$ and identify a region whose area is $d_n - d_{n+1}$.)

**3.** Use an appropriate theorem from the text to show that the sequence $\{d_n\}$ is convergent. Letting $\gamma = \lim\limits_{n\to\infty} d_n$, this limit is called **Euler's constant**. It is important in many applications throughout various areas of mathematics, and like other famous constants (including $\pi$ and $e$) can be approximated with great precision using modern computing power. Surprisingly, however, it is not yet known whether $\gamma$ is rational or irrational!

**4.** Use the convergence of $\{d_n\}$ to prove that the sequence $a_n = \sum_{i=n}^{2n} \frac{1}{i}$ converges and find its limit.

**5.** Use a computer algebra system to approximate $\gamma$, accurate to the first 10 decimal places.

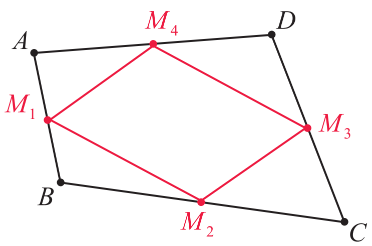
**6.** Use the approximate value of $\gamma$ found in Question 5 to estimate $s_n$, rounded to 5 decimal places, for **a.** $n = 10{,}000$ and **b.** $n = 2{,}000{,}000$. Compare the latter estimate with the answer for Exercise 125b of the Chapter Review.

# Chapter 11 Project

In this project, we are going to use vectors to prove an interesting property of quadrilaterals. In fact, the result is general enough that our quadrilateral doesn't have to be planar, in other words, its vertices do not have to lie in the same plane!

**1.** Let $A$, $B$, $C$, $D$ be four points in $\mathbb{R}^3$, with $M_1$, $M_2$, $M_3$, and $M_4$ being the midpoints of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Consider the vector $\overrightarrow{M_1M_2}$ and show that

$$\overrightarrow{M_1M_2} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right).$$




**2.** Prove a statement analogous to the one in Question 1 for the vector $\overrightarrow{M_3M_4}$ and show that

$$\overrightarrow{M_1M_2} = -\overrightarrow{M_3M_4}.$$

**3.** Using the results of Questions 1 and 2, argue that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram.

**4.** Explain why the proof of Question 3 does not require that the points $A$, $B$, $C$, and $D$ lie in the same plane.

**5.** Use vectors in the three-dimensional coordinate system to prove the statement of Question 3: If $ABCD$ is a (not necessarily planar) quadrilateral in $\mathbb{R}^3$, then the midpoints of its sides determine a parallelogram. (**Hint:** To simplify your calculations, you can assume that three of the vertices lie in the same coordinate plane with one of them, say $A$, located at the origin, and an adjacent vertex, such as $B$, lying on a coordinate axis.)

# Chapter 12 Project

In this project you will use vector functions to develop a simple model for 3-point basketball shots. To keep the model simple, we will be ignoring air resistance, friction, and other forces. Furthermore, by "scoring," we will mean that the ball falls straight into the basket on its way downward (i.e., we will ignore the possibility of the ball bouncing in off the backboard, or any energy losses as a result of spins, etc.). For further studies, or for more refined models, the interested student should consult resources such as John Fontanella's book, *The Physics of Basketball*.

1. A basketball player is attempting a 3-pointer from a horizontal distance of 24.7 feet. He is releasing the ball from 7 feet above ground level, aimed directly toward the basket at an angle of elevation of 45°, with an initial velocity of $v_0$. Supposing that the player stands at the origin and the basket is in the positive $y$-direction, use the three-dimensional coordinate system to find a vector function describing the position of the ball after release as a function of time. (Assume one unit on each axis corresponds to a distance of 1 foot.)

2. Use your answer to Question 1 to verify that the basketball's trajectory is a parabola.

3. Assuming a standard hoop height of 10 feet, find the initial speed for the ball that ensures that the player described in Question 1 scores.

4. Use your answer from Question 3 to find the necessary initial velocity vector for the basketball if the player is to score from the same spot (i.e., the origin) but this time shooting while running along the line $y = x$ at a speed of 10 mph in the positive direction.

5. Find a formula for and graph the required initial speed as a function of the angle of elevation over the interval $(0, \pi/2)$ if the player is to score (assuming the same spot and release height as in Question 1).

6. Generalizing your work on Question 5, find a formula for the initial speed of a successful shot if the player stands $d$ feet from the hoop and shoots at an angle $\alpha$ upward from horizontal, with a release height of $h$ feet.

# Chapter 13 Project

In this project you will use your experience with partial derivatives and differentials to learn how to solve an important class of differential equations, called **exact equations**. Ordinary differential equations of this type are noted for their widespread applications in physics and engineering. (See Section 8.1 for the definitions of differential equation and solution. Other than the basic definitions, this project does not directly rely on, and can be considered independently of Chapter 8.)

**1.** Suppose that the first-order partial derivatives of the function $z = f(x,y)$ are both continuous on a region $R$. If $c$ is a constant and $y = y(x)$ is defined implicitly by the equation $f(x,y) = c$, show that $y$ solves the differential equation

$$f_y(x,y)\cdot y' = -f_x(x,y).$$

**2.** Now consider a differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (1)$$

and assume that there is a two-variable function $f(x,y)$ such that

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

(such a differential equation is called *exact*, while $f(x,y)$ is called a *potential function*). Use your answer to Question 1 to show that the set of level curves $f(x,y) = C,\ C \in \mathbb{R}$ form a family of solutions of the differential equation (1).

**3.** Suppose that $M(x,y)$ and $N(x,y)$, as well as their first-order partial derivatives, are continuous on an open region $R$. Show that a necessary condition for equation (1) to be exact is the following equality.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

(**Note:** If we require a bit more of $R$, the above condition is also sufficient for exactness, a statement we will not rigorously prove here, but the construction of a potential function under the stated conditions is outlined in Questions 5 and 6.)

**4.** Use Question 3 to determine which of the following equations is exact.

**a.** $\left(2x + ye^{xy}\right)dx + \left(xe^{xy} - 1\right)dy = 0$

**b.** $\left(3x^2 y - \dfrac{1}{\sqrt{x}}\right)dx + \left(x^3 - \sqrt{x}\right)dy = 0$

**5.** Explain why the potential function $f$ of an exact equation must satisfy

$$f(x,y) = \int M(x,y)\,dx + g(y),$$

where $g$ is some function of the variable $y$.

**6.** Show that if the equation (1) is exact, then the equality

$$N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$$

must hold.

**7.** Use Questions 5 and 6 to solve the equation $2\left(x - y^2\right)dx + y(9y - 4x)\,dy = 0$ by determining its potential function $f(x,y)$ and identifying the family of solutions as $f(x,y) = C$. (**Hint:** After identifying $M(x,y)$ and $N(x,y)$, use Question 5 to obtain a tentative formula for $f(x,y)$, then use Question 6 to determine the unknown function $g(y)$.)

**8.** Verify that the equation $(2 + x)\,y\,dx + 2x\,dy = 0$ becomes exact after multiplying by the integrating factor $I(x,y) = xye^x$. Solve the resulting equation.

# Chapter 14 Project

In this project you will be able to take advantage of useful coordinate transformations to evaluate multiple integrals on ellipses and ellipsoids that would be much more challenging in the Cartesian coordinate system.

1. Find the Jacobian of the coordinate transformation $T(r,\theta)$ defined by $x = ar\cos\theta$ and $y = br\sin\theta$, where $a$, $b > 0$.

2. Use double integration along with the coordinate transformation in Question 1 to arrive at the formula for the area $A$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Find the Jacobian of the transformation to "ellipsoidal coordinates" $T_e(\rho,\theta,\phi)$ defined by $x = a\rho\sin\phi\cos\theta$, $y = b\rho\sin\phi\sin\theta$, and $z = c\rho\cos\phi$, where $a$, $b$, $c > 0$.

4. Use double integration along with the coordinate transformation of Question 3 to arrive at the formula for the volume $V$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

5. Use ellipsoidal coordinates to find the center of mass of the upper ellipsoid $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, assuming constant density.

6. Find the mass and the center of mass of the semiellipsoid of Question 5 in the case that the density at any point is proportional to the distance from the $xy$-plane.

7. Use a computer algebra system and ellipsoidal coordinates to find the second moments and radii of gyration for the solid of Question 5. Express the second moments in terms of the mass $m$ of the semiellipsoid.

8. Use a computer algebra system and ellipsoidal coordinates to find the second moments and radii of gyration for the solid of Question 6. As in the previous problem, express the second moments in terms of the mass $m$ of the semiellipsoid.

# Chapter 15 Project

Recall from Section 15.7 that if $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ so that $\nabla \times \mathbf{F} = \mathbf{0}$ (such vector fields are called curl-free) on an open, simply connected domain in space, then $\mathbf{F}$ is conservative, that is, there is a scalar potential $f$ so that $\nabla f = \mathbf{F}$. On the other hand, it can be shown that if $\mathbf{F}$ is divergence-free, that is, if $\nabla \cdot \mathbf{F} = 0$, then there is a vector field $\mathbf{P}$ such that $\nabla \times \mathbf{P} = \mathbf{F}$ (such a vector field is called a *vector potential* for $\mathbf{F}$). In this project you will discover a way of finding a vector potential for a given divergence-free vector field $\mathbf{F}$.

**1.** Suppose

$$\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$

and

$$\mathbf{P}(x,y,z) = \langle P_1(x,y,z), P_2(x,y,z), P_3(x,y,z) \rangle$$

are vector fields so that $\nabla \times \mathbf{P} = \mathbf{F}$; that is, $\mathbf{P}$ is a vector potential for $\mathbf{F}$. Show that for any differentiable scalar field $f$, $\nabla \times (\mathbf{P} + \nabla f) = \mathbf{F}$; that is, $\mathbf{P} + \nabla f$ is another vector potential for $\mathbf{F}$. (**Hint:** See Exercise 41 of Section 15.4.)

**2.** If $f$ is any scalar field such that $\frac{\partial f}{\partial x} = -P_1$, show that if we define $\hat{P} = P + \nabla f$, then $\hat{P}_1 = 0$.

**3.** Use Questions 1 and 2 to argue that if the vector field $\mathbf{F}$ has a vector potential $\mathbf{P}$, then it has one whose first component is zero. In other words, we may assume throughout our discussion that $\mathbf{P} = \langle 0, P_2, P_3 \rangle$.

In Questions 4–6, you will be guided to show that given a divergence-free vector field $\mathbf{F}$, it is possible and fairly straightforward to find a vector potential of the form described in Question 3.

**4.** Assume that

$$\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$

is a vector field such that $\nabla \cdot \mathbf{F} = 0$, and $\mathbf{P}$ is any vector field of the form $\mathbf{P} = \langle 0, P_2, P_3 \rangle$. Show that $\mathbf{P}$ is a vector potential for $\mathbf{F}$ if the following equalities hold:

$$\frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} = F_1 \qquad -\frac{\partial P_3}{\partial x} = F_2 \qquad \frac{\partial P_2}{\partial x} = F_3$$

**5.** For the vector field $\mathbf{F}$ in Question 4, define $P_2(x,y,z) = \int_{x_0}^{x} F_3(t,y,z)\,dt + C_2(y,z)$ and $P_3(x,y,z) = -\int_{x_0}^{x} F_2(t,y,z)\,dt + C_3(y,z)$, where $x_0$ is an arbitrary starting value and $C_2$ and $C_3$ are arbitrary functions of the variables $y$ and $z$. Show that $\mathbf{P}(x,y,z) = \langle 0, P_2(x,y,z), P_3(x,y,z) \rangle$ satisfies the last two equations in Question 4.

**6.** Show that in Question 5, it is always possible to choose $C_2(y,z)$ and $C_3(y,z)$ to satisfy $\frac{\partial P_3}{\partial y} - \frac{\partial P_2}{\partial z} = F_1$, and conclude that $\mathbf{P}(x,y,z) = \langle 0, P_2(x,y,z), P_3(x,y,z) \rangle$ will then be a vector potential for $\mathbf{F}$. (**Hint:** Use the fact that $\nabla \cdot \mathbf{F} = 0$.)

**7.** Show that the vector field

$$\mathbf{F}(x,y,z) = \langle 2x^2yz, -2xy^2z, x^2y \rangle$$

is divergence-free, and follow the steps outlined in Questions 5 and 6 to find a vector potential for $\mathbf{F}$. (Answers may vary.)