Example 6: Using Trigonometric Substitutions

Use the substitution $\sin\theta = \frac{x}{2}$ to write $\sqrt{4-x^2}$ as a trigonometric expression. Assume $0 \le \theta \le \frac{\pi}{2}$.

Solution

Although it is not necessary for the task at hand, a diagram motivating the substitution may be helpful. The triangle in Figure 3 illustrates the geometric relation between θ and the various algebraic expressions.

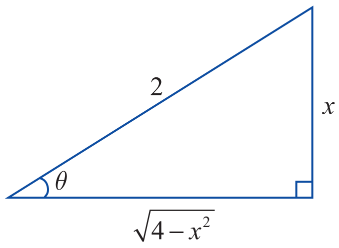

FIGURE 3

The suggested substitution can be rewritten as $x = 2\sin\theta$, and so we obtain the following.

$$\begin{aligned}\sqrt{4-x^2} &= \sqrt{4-(2\sin\theta)^2}\\ &= \sqrt{4-4\sin^2\theta}\\ &= 2\sqrt{1-\sin^2\theta}\\ &= 2\sqrt{\cos^2\theta}\\ &= 2\cos\theta\end{aligned}$$

We can write $2\cos\theta$ instead of $2|\cos\theta|$ since the restriction $0 \le \theta \le \frac{\pi}{2}$ means $\cos\theta \ge 0$.

9.1 EXERCISES

PRACTICE

Use trigonometric identities to simplify the expressions. There may be more than one correct answer. See Examples 1 and 2.

1. $\tan x \csc x$
2. $\dfrac{1}{\tan^2\theta + 1}$
3. $\dfrac{\tan t}{\sec t}$
4. $\cot^2 x - \cot^2 x \cos^2 x$
5. $\sin(-x)\tan x$
6. $\dfrac{1}{\sec^2 x} + \sin x \cos\left(\dfrac{\pi}{2} - x\right)$
7. $\sin(\alpha + 2\pi)\sec\alpha$
8. $\sin t(\csc t - \sin t)$
9. $\cos y(1 + \tan^2 y)$
10. $\dfrac{1}{\cos x \csc(-x)}$
11. $\dfrac{1-\tan^2 x}{\cot^2 x - 1}$
12. $\dfrac{\sin\beta \tan\left(\dfrac{\pi}{2} - \beta\right)}{\cos\beta}$

Use the suggested substitution to rewrite the given expression as a trigonometric expression. See Example 6. Assume $0 \le \theta \le \frac{\pi}{2}$.

13. $\sqrt{x^2+1}, \quad x=\tan\theta$

14. $\sqrt{x^2-16}, \quad x=4\sec\theta$

15. $\sqrt{9-x^2}, \quad \cos\theta=\frac{x}{3}$

16. $\sqrt{4x^2+100}, \quad \cot\theta=\frac{x}{5}$

17. $\sqrt{64-x^2}, \quad x=8\sin\theta$

18. $\sqrt{x^2-4}, \quad x=2\csc\theta$

19. $\sqrt{x^2+25}, \quad \tan\theta=\frac{x}{5}$

20. $\sqrt{144-9x^2}, \quad x=4\cos\theta$

WRITING & THINKING

Verify the following trigonometric identities. See Examples 3, 4, and 5.

21. $(1-\cos\theta)(1+\cos\theta)=\sin^2\theta$

22. $\csc x-\sin x=\cos x\cot x$

23. $\sec^2 y-\tan^2 y=\sec y\cos(-y)$

24. $(1-\sin\beta)(\sec\beta\tan\beta)=\frac{\sin\beta}{1+\sin\beta}$

25. $\frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)}=\cot x$

26. $\frac{\sec^2\theta}{\tan\theta}=\sec\theta\csc\theta$

27. $\frac{1}{\tan x}+\tan x=\frac{\sec^2 x}{\tan x}$

28. $\sin^2 t+\sin^2\left(\frac{\pi}{2}-t\right)=1$

29. $\frac{1}{\sin(\theta+2\pi)+1}+\frac{1}{\csc(\theta+2\pi)+1}=1$

30. $3+\cot^2\alpha=2+\csc^2\alpha$

31. $\sin^2 x-\sin^4 x=\cos^2(-x)-\cos^4(-x)$

32. $\cot\left(\frac{\pi}{2}-\beta\right)\cot\beta=1$

33. $\frac{\cos\left(\frac{\pi}{2}-\alpha\right)}{\csc\alpha}-1=\sin\alpha\cot(-\alpha)\cos(-\alpha)$

Show how the identities below follow from the first Pythagorean identity.

34. $\tan^2 x+1=\sec^2 x$

35. $1+\cot^2 x=\csc^2 x$