13.4 EXERCISES

PRACTICE

Find S_{k+1} for the given S_k.

1. $S_k = \dfrac{1}{3(k+2)}$

2. $S_k = \dfrac{k^2}{k(k-1)}$

3. $S_k = \dfrac{k(k+1)(2k+1)}{4}$

4. $S_k = \dfrac{1}{(2k-1)(2k+1)}$

Use the Principle of Mathematical Induction to prove the following statements.

5. $1+2+3+4+\cdots+n = \dfrac{n(n+1)}{2}$

6. $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$

7. $2+4+6+8+\cdots+2n = n(n+1)$

8. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

9. $4^0+4^1+4^2+\cdots+4^{n-1} = \dfrac{4^n-1}{3}$

10. $2^n > n^2$ for all $n \ge 5$.

11. $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

12. $5^0 + 5^1 + 5^2 + \cdots + 5^{n-1} = \frac{5^n - 1}{4}$

13. $5 + 10 + 15 + \cdots + 5n = \frac{5n(n+1)}{2}$

14. $n^2 \geq 100n$ for all $n \geq 100$.

15. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$

16. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$

17. $1 + 4 + 7 + 10 + \cdots + (3n-2) = \frac{n}{2}(3n-1)$

18. $-2 - 3 - 4 - \cdots - (n+1) = -\frac{1}{2}(n^2 + 3n)$

19. $3^n > 2n+1$ for all $n \geq 2$.

20. $2^n > n$, for all $n \geq 1$

21. $1^3+2^3+3^3+4^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

22. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

23. If $a>1$, then $a^n>1$.

24. $2^n>4n$ for all $n\geq 5$.

25. $1^4+2^4+3^4+4^4+\cdots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

26. $1^5+2^5+3^5+4^5+\cdots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

27. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$ for all $n \geq 2$.

28. $1+3+5+7+\cdots+(2n-1)=n^2$

Use the Principle of Mathematical Induction to prove the given properties. (Assume *m* and *n* are natural numbers and *a*, *b*, and *x* are real numbers.)

29. $(ab)^n = a^n b^n$ (Assume a and b are constants.)

30. $(a^m)^n = a^{mn}$ (Assume a and m are constants.)

31. If $x_1 > 0, x_2 > 0, ..., x_n > 0,$ then
$\ln(x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_n$.

32. 5 is a factor of $\left(2^{2n-1}+3^{2n-1}\right)$.

33. 64 is a factor of $(9^n - 8n - 1)$ for all $n \geq 2$.

34. 3 is a factor of $(n^3 + 3n^2 + 2n)$.

35. $n^3 - n + 3$ is divisible by 3.

36. $5^n - 1$ is divisible by 4.

37. $n(n+1)(n+2)$ is divisible by 6.

APPLICATIONS

38. In the 19th century a mathematical puzzle was published telling of a mythical monastery in Benares, India with three crystal towers holding 64 disks made of gold. The disks are each of a different size and have holes in the middle so that they slide over the towers and sit in a stack with the largest on the bottom and the smallest on the top. The monks of the monastery were instructed to move all of the disks to the third tower following these three rules:

- Each disk sits over a tower except when it is being moved.
- No disk may ever rest on a smaller disk.
- Only one disk at a time may be moved.

According to the puzzle, when the monks complete their task, the world would end! To move n disks requires $H(n) = 2^n - 1$ moves. Prove this is true through mathematical induction.

39. If there are n people in a room, and every person shakes hands with every other person exactly once, then exactly $\frac{n(n-1)}{2}$ handshakes will occur. Prove this is true through mathematical induction.

40. Any monetary value of 4 cents or higher can be composed of twopence (a British two-cent coin) and nickels. Your basic step would be

$$4 \text{ cents} = \text{twopence} + \text{twopence}.$$

Use the fact that $k = 2t + 5n$ where k is the total monetary value, t is the number of twopence, and n is the number of nickels, to prove $P(k+1)$. (**Hint:** There are 3 induction steps to prove.)

WRITING & THINKING

41. What is wrong with this "proof" by induction?

Proposition: All horses are the same color. (In any set of n horses, all horses are the same color.)

Basic Step: If you have only one horse in a group, then all of the horses in that group have the same color.

Induction Step: Assume that in any group of n horses, all horses are the same color. Now take any group of $n + 1$ horses. Remove the first horse from this group and the remaining n horses must be of the same color because of the hypothesis. Now replace the first horse and remove the last horse. Once again, the remaining n horses must be the same color because of the hypothesis. Since the two groups overlap, all $n + 1$ horses must be the same color.

Thus by induction, any group of n horses are the same color.