Solution

Write out the first few terms of the series and determine the common ratio.

$$\sum_{k=1}^{\infty} 3\left(\frac{1}{10}\right)^k = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots$$

We can see that $a_1 = \frac{3}{10}$ and the common ratio is $r = \frac{1}{10}$. Substitute these values into the formula and simplify.

$$\begin{aligned} S &= \frac{a_1}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} \\ &= \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{10} \cdot \frac{10}{9} \\ &= \frac{1}{3} \end{aligned}$$

Now work margin exercise 10.

The sum calculated in Example 10 illustrates the relationship between geometric series and the decimal system. Notice that the sum can also be written with decimal values.

$$\begin{aligned} \sum_{k=1}^{\infty} 3\left(\frac{1}{10}\right)^k &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots \\ &= 0.3 + 0.03 + 0.003 + 0.0003 + \cdots \\ &= 0.33333\ldots \\ &= 0.\overline{3} \end{aligned}$$

This confirms that $0.\overline{3}$ is the decimal representation of the fraction $\frac{1}{3}$.

Margin Exercise Answers

1. Geometric **2.** Not geometric **3.** $a_6 = 224$ **4.** $\frac{2}{243}$ **5.** $a_1 = 0.01$ and $r = 2$ or $a_1 = 0.01$ and $r = -2$
6. $\frac{255}{256}$ **7.** $\frac{-\sqrt{2}\left(1+4\sqrt{2}\right)}{1+\sqrt{2}}$ **8.** \$49,139.99 **9. a.** 4 **b.** $\frac{16}{5}$ **10.** 1

12.4 Exercises

Concept Check

Fill-in-the-Blank. Complete each sentence using information found in this section.

1. Any two consecutive terms in a geometric sequence have the _______ ratio.
2. For a geometric sequence, the letter r denotes the ___________.
3. The formula for the general term of a geometric sequence is ________.
4. The n^{th} partial sum S_n is the sum of the first ___ terms of a geometric sequence.
5. The indicated sum of all terms in a sequence is called a/an ______ series.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (**Note:** There may be more than one acceptable change.)

6. Another name for a geometric sequence is *geometric progression*.

7. To determine whether a sequence is geometric, find the difference of consecutive terms and determine if that difference is constant.

Practice

Determine whether each sequence is geometric. If the sequence is geometric, find the common ratio and the general n^{th} term.

1. $2, 4, 6, 8, \ldots$

2. $\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \ldots$

3. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

4. $5, 9, 13, 17, \ldots$

5. $\frac{32}{27}, \frac{4}{9}, \frac{1}{6}, \frac{1}{16}, \ldots$

6. $18, 12, 8, \frac{16}{3}, \ldots$

7. $\frac{14}{3}, \frac{2}{3}, \frac{2}{15}, \frac{2}{45}, \ldots$

8. $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots$

9. $48, -12, 3, -\frac{3}{4}, \ldots$

10. $4, -8, 12, -16, \ldots$

Write the first four terms of each sequence, then determine whether the sequence is geometric.

11. $\left\{(-3)^{n+1}\right\}$

12. $\left\{3\left(\frac{2}{5}\right)^n\right\}$

13. $\left\{\frac{2}{3}n\right\}$

14. $\left\{(-1)^{n+1}\left(\frac{2}{7}\right)^n\right\}$

15. $\left\{2\left(-\frac{4}{5}\right)^n\right\}$

16. $\left\{1+\frac{1}{2^n}\right\}$

17. $\left\{3(2)^{\frac{n}{2}}\right\}$

18. $\left\{\frac{n^2+1}{n}\right\}$

19. $\left\{(-1)^{n-1}(0.3)^n\right\}$

20. $\left\{6(10)^{1-n}\right\}$

Use the given information to find the general form $\{a_n\}$ of each geometric sequence.

21. $a_1 = 3,\ r = 2$

22. $a_1 = -2,\ r = \frac{1}{5}$

23. $a_1 = \frac{1}{3},\ r = -\frac{1}{2}$

24. $a_1 = 5,\ r = \sqrt{2}$

25. $a_3 = 2,\ a_5 = 4,\ r > 0$

26. $a_4 = 19,\ a_5 = 57$

27. $a_2 = 1,\ a_4 = 9,\ r > 0$

28. $a_2 = 5,\ a_5 = \frac{5}{8}$

29. $a_3 = -\frac{45}{16},\ r = -\frac{3}{4}$

30. $a_4 = 54,\ r = 3$

Assume each sequence is geometric. Find the indicated value.

31. $a_1 = -32,\ a_6 = 1.$ Find a_8.

32. $a_1 = 20,\ a_6 = \frac{5}{8}.$ Find a_7.

33. $a_1 = 18,\ a_7 = \frac{128}{81}.$ Find a_5.

34. $a_1 = -3,\ a_5 = -48.$ Find a_7.

35. $a_3 = \frac{1}{2},\ a_7 = \frac{1}{32}.$ Find a_4.

36. $a_5 = 48,\ a_8 = -384.$ Find a_9.

37. $a_1 = -2,\ r = \frac{2}{3},\ a_n = -\frac{16}{27}.$ Find n.

38. $a_1 = \frac{1}{9},\ r = \frac{3}{2},\ a_n = \frac{27}{32}.$ Find n.

Use the formulas for partial sums of geometric sequences and sums of geometric series to calculate the sums.

39. $3 + 9 + 27 + 81 + 243$

40. $-2 + 4 - 8 + 16$

41. $8 + 4 + 2 + \cdots + \frac{1}{64}$

42. $3 + 12 + 48 + \cdots + 3072$

43. $\sum_{k=1}^{3} -3\left(\frac{3}{4}\right)^k$

44. $\sum_{k=1}^{6} \left(-\frac{5}{3}\right)\left(\frac{1}{2}\right)^k$

45. $\sum_{k=1}^{5} \left(\frac{2}{3}\right)^k$

46. $\sum_{k=1}^{6} \left(\frac{1}{3}\right)^k$

47. $\sum_{k=4}^{7} 5\left(\frac{1}{2}\right)^k$

48. $\sum_{k=3}^{6} -7\left(\frac{3}{2}\right)^k$

49. $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1}$

50. $\sum_{k=1}^{\infty} \left(\frac{5}{8}\right)^{k-1}$

51. $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$

52. $\sum_{k=1}^{\infty} \left(-\frac{2}{5}\right)^k$

53. $0.\overline{4}$

54. $0.\overline{6}$

55. $0.\overline{36}$

56. $0.\overline{81}$

Applications

Solve.

57. When Henry was born, his grandmother deposited \$10,000 in a trust account bearing 5% interest compounded annually. The account was set up for Henry to access the money when he turned 21. How much money was in the account on his 21st birthday?

58. An automobile that costs \$18,500 when purchased depreciates at a rate of 20% of its value each year. What is its value after 4 years?

59. A fish is in a tank with 20 liters of river water. To acclimate the fish to a new environment, 4 liters of river water are drained off and replaced with aquarium water. The next day, 4 liters of the mixture are drained off and replaced with aquarium water. This process is continued until six drain-offs and replacements have been made. How much aquarium water is in the final mixture?

60. Suppose $1200 is deposited in a savings account each year for 8 years. If interest is compounded annually at 6%, what would be the value of the account at the end of 8 years?

61. Kathleen purchases a $1000 certificate of deposit (CD) each year for 10 years. If the annual interest rate on each CD is 4.5%, what will be the total value of these CDs after 10 years?

62. A substance decays at a rate of $\frac{2}{5}$ of its weight per day. How much of the substance will be present after 4 days if initially there are 500 grams?

63. A ball rebounds to a height that is $\frac{3}{4}$ of its original height. How high will it rise after the fourth bounce if it is dropped from a height of 24 meters?

Writing & Thinking

64. Graph the first 8 partial sums of each geometric series as points to show how the sum of the series approaches a certain value. Show this value as a horizontal line on the graph.

a. $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$

b. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3^k}$

65. Consider the infinite series $4\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$. Write out several (at least 10 to 15) of the partial sums and their values until you can identify the number the partial sums *seem* to be approaching. What is this number?

66. Explain why there is no formula for finding the sum of an infinite geometric series when $|r|>1$.