Notation and Terminology

Exponents

$$\underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} = a^n$$

exponent

base

Fractions

$\frac{a}{b}$ — a ← numerator, b ← denominator

Least Common Multiple (LCM)

Given a set of counting numbers, the smallest number that is a multiple of each of these numbers.

Greatest Common Factor (GCF)

Given a set of integers, the largest integer that is a factor (or divisor) of all of the integers.

Types of Numbers

Natural Numbers (Counting Numbers):

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$

Whole Numbers: $\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

Rational Numbers: A number that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Irrational Numbers: A number that can be written as an infinite nonrepeating decimal.

Real Numbers: All rational and irrational numbers.

Complex Numbers: All real numbers and the even roots of negative numbers. The standard form of a complex number is $a + bi$, where a and b are real numbers, a is called the real part and b is called the imaginary part.

Absolute Value

$|a|$ The distance a real number a is from 0.

Equality and Inequality Symbols

- $=$ "is equal to"
- $\neq$ "is not equal to"
- $<$ "is less than"
- $>$ "is greater than"
- $\leq$ "is less than or equal to"
- $\geq$ "is greater than or equal to"

Sets

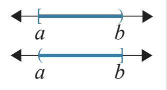
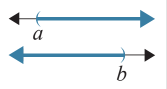
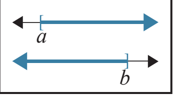
The **empty set** or **null set** (symbolized $\varnothing$ or $\{\ \}$): A set with no elements.

The **union** of two (or more) sets (symbolized $\cup$): The set of all elements that belong to either one set or the other set or to both sets.

The **intersection** of two (or more) sets (symbolized $\cap$): The set of all elements that belong to both sets.

The word **or** is used to indicate union and the word **and** is used to indicate intersection.

Algebraic and Interval Notation for Intervals

Type of Interval	Algebraic Notation	Interval Notation	Graph
Open Interval	$a < x < b$	(a, b)	
Closed Interval	$a \leq x \leq b$	$[a, b]$	
Half-Open Interval	$\begin{cases} a \leq x < b \\ a < x \leq b \end{cases}$	$[a, b)$ $(a, b]$	
Open Interval	$\begin{cases} x > a \\ x < b \end{cases}$	(a, ∞) $(-\infty, b)$	
Half-Open Interval	$\begin{cases} x \geq a \\ x \leq b \end{cases}$	$[a, \infty)$ $(-\infty, b]$	

Radicals

The symbol $\sqrt{}$ is called a **radical sign**.
The number under the radical sign is called the **radicand**.

The complete expression, such as $\sqrt{64}$, is called a **radical** or **radical expression**.

In a cube root expression $\sqrt[3]{a}$, the number 3 is called the index. In a square root expression such as $\sqrt{a}$, the index is understood to be 2 and is not written.

The Imaginary Number *i*

$i = \sqrt{-1}$ and $i^2 = \left(\sqrt{-1}\right)^2 = -1$

n Factorial (*n*!)

For any positive integer n, **the factorial of *n***, denoted as $n!$, is the product of all positive integers from n through 1.

$n! = n(n-1)(n-2)\cdots(3)(2)(1)$

Principles and Properties

Properties of Addition and Multiplication

Property	Addition	Multiplication
Commutative Property	$a+b=b+a$	$ab=ba$
Associative Property	$(a+b)+c=a+(b+c)$	$a(bc)=(ab)c$
Identity	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Inverse	$a+(-a)=0$	$a\cdot\frac{1}{a}=1\ (a\neq 0)$

Zero-Factor Law: $a\cdot 0=0\cdot a=0$

Distributive Property: $a(b+c)=a\cdot b+a\cdot c$

Addition (or Subtraction) Principle of Equality

$A=B$, $A+C=B+C$, and $A-C=B-C$ have the same solutions (where A, B, and C are algebraic expressions).

Multiplication (or Division) Principle of Equality

$A=B$, $AC=BC$, and $\frac{A}{C}=\frac{B}{C}$ have the same solutions (where A and B are algebraic expressions and C is any nonzero constant, $C\neq 0$).

Properties of Exponents

For nonzero real numbers a and b and integers m and n:

The exponent 1	$a=a^1$
The exponent 0	$a^0=1$
The product rule	$a^m\cdot a^n=a^{m+n}$
The quotient rule	$\frac{a^m}{a^n}=a^{m-n}$
Negative exponents	$a^{-n}=\frac{1}{a^n}$
Power rule	$(a^m)^n=a^{mn}$
Power of a product	$(ab)^n=a^nb^n$
Power of a quotient	$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$

Zero-Factor Law

If a and b are real numbers, and $a\cdot b=0$, then $a=0$ or $b=0$ or both.

Properties of Rational Expressions

If $\frac{P}{Q}$ is a rational expression and P, Q, R, S, and K are polynomials where $Q, R, S, K\neq 0$, then

The Fundamental Principle	$\frac{P}{Q}=\frac{P\cdot K}{Q\cdot K}$
Multiplication	$\frac{P}{Q}\cdot\frac{R}{S}=\frac{P\cdot R}{Q\cdot S}$
Division	$\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}$
Addition	$\frac{P}{Q}+\frac{R}{Q}=\frac{P+R}{Q}$
Subtraction	$\frac{P}{Q}-\frac{R}{Q}=\frac{P-R}{Q}$

Properties of Radicals

If a and b are positive real numbers, n is a positive integer, m is any integer, and $\sqrt[n]{a}$ is a real number then

1. $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $\sqrt[n]{a}=a^{\frac{1}{n}}$
4. $a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^m=(a^m)^{\frac{1}{n}}$

   or, in radical notation,

   $a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m=\sqrt[n]{a^m}$

Properties of Logarithms

For $b>0$, $b\neq 1$, $x, y>0$, and any real number r,

1. $\log_b 1=0$
2. $\log_b b=1$
3. $x=b^{\log_b x}$
4. $\log_b b^x=x$
5. $\log_b xy=\log_b x+\log_b y$ The product rule
6. $\log_b\frac{x}{y}=\log_b x-\log_b y$ The quotient rule
7. $\log_b x^r=r\cdot\log_b x$ The power rule

Properties of Equations with Exponents and Logarithms

For $b>0$, $b\neq 1$,

1. If $b^x=b^y$, then $x=y$.
2. If $x=y$, then $b^x=b^y$.
3. If $\log_b x=\log_b y$, then $x=y$ ($x>0$ and $y>0$).
4. If $x=y$, then $\log_b x=\log_b y$ ($x>0$ and $y>0$).

Equations and Inequalities

Linear Equation in x (First-Degree Equation in x)

$ax + b = c$, where a, b, and c are real numbers and $a \neq 0$.

Types of Equations and their Solutions

Conditional: Finite Number of Solutions

Identity: Infinite Number of Solutions

Contradiction: No Solution

Linear Inequalities

Linear inequalities have the following forms where a, b, and c are real numbers and $a \neq 0$:

$ax + b < c$ and $ax + b \leq c$

$ax + b > c$ and $ax + b \geq c$

Compound Inequalities

The inequalities $c < ax + b < d$ and $c \leq ax + b \leq d$ are called **compound linear inequalities**. (This includes $c < ax + b \leq d$ and $c \leq ax + b < d$ as well.)

Absolute Value Equations

For statements 1 and 2, $c > 0$:

1. If $|x| = c$, then $x = c$ or $x = -c$.

2. If $|ax + b| = c$, then $ax + b = c$ or $ax + b = -c$.

3. If $|a| = |b|$, then either $a = b$ or $a = -b$.

4. If $|ax + b| = |cx + d|$, then either $ax + b = cx + d$ or $ax + b = -(cx + d)$.

Absolute Value Inequalities

For $c > 0$:

1. If $|x| < c$, then $-c < x < c$.

2. If $|ax + b| < c$, then $-c < ax + b < c$.

3. If $|x| > c$, then $x < -c$ **or** $x > c$.

4. If $|ax + b| > c$, then $ax + b < -c$ **or** $ax + b > c$.

(These statements hold true for $\leq$ and $\geq$ as well.)

Quadratic Equation

An equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

Quadratic Formula

The solutions of the general quadratic equation

$ax^2 + bx + c = 0$, where $a \neq 0$, are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The Discriminant

The expression $b^2 - 4ac$, the part of the quadratic formula that lies under the radical sign, is called the **discriminant**.

If $b^2 - 4ac > 0$, there are two real solutions.

If $b^2 - 4ac = 0$, there is one real solution, $x = -\dfrac{b}{2a}$.

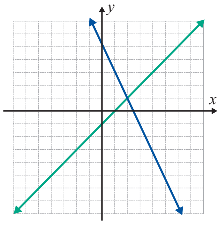
If $b^2 - 4ac < 0$, there are two nonreal solutions.

Systems of Linear Equations

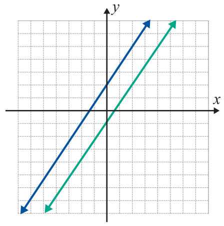
Systems of Linear Equations (Two Variables)

The system is...

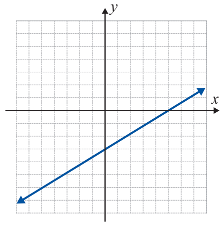
consistent, and
the equations are **independent**.
(One solution)



inconsistent, and
the equations are **independent**.
(No solution)



consistent, and
the equations are **dependent**.
(Infinite number of solutions)



Functions

Function, Relation, Domain, and Range

A relation is a set of ordered pairs of real numbers.

The domain ***D*** of a relation is the set of all first coordinates in the relation.

The range ***R*** of a relation is the set of all second coordinates in the relation.

A function is a relation in which each domain element has exactly one corresponding range element.

One-to-One Functions

A function is a one-to-one function if for each value of y in the range there is only one corresponding value of x in the domain.

Algebraic Operations with Functions

1. $(f+g)(x) = f(x) + g(x)$

2. $(f-g)(x) = f(x) - g(x)$

3. $(f \cdot g)(x) = f(x) \cdot g(x)$

4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

5. $(f \circ g)(x) = f(g(x))$

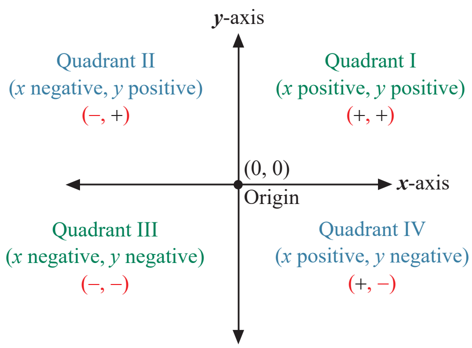
Inverse Functions

If f is a one-to-one function with ordered pairs of the form (x, y), then its inverse function, denoted as f^{-1}, is also a one-to-one function with ordered pairs of the form (y, x).

If f and g are one-to-one functions and $f(g(x)) = x$ for all x in D_g and $g(f(x)) = x$ for all x in D_f, then f and g are inverse functions.

Graphs of Functions

The Cartesian Coordinate System



Linear Functions (Lines)

Standard form:

$Ax + By = C$ Where A and B do not both equal 0

Slope of a line:

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ Where $x_1 \neq x_2$

Slope-intercept form:

$y = mx + b$ With slope m and y-intercept $(0, b)$

Point-slope form:

$y - y_1 = m(x - x_1)$ With slope m and point (x_1, y_1) on the line

Horizontal line, slope 0: $y = b$

Vertical line, undefined slope: $x = a$

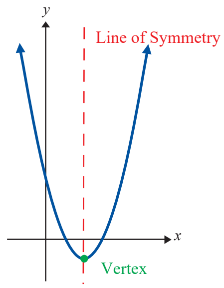
Parallel lines have the same slope.

Perpendicular lines have slopes that are negative reciprocals of each other.

Quadratic Functions (Parabolas)

Parabolas of the form $y = ax^2 + bx + c$:

1. Vertex: $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.
2. Line of Symmetry: $x = -\dfrac{b}{2a}$



Parabolas of the form $y = a(x-h)^2 + k$:

1. Vertex: (h, k)
2. Line of Symmetry: $x = h$
3. The graph is a horizontal shift of h units and a vertical shift of k units of the graph of $y = ax^2$.

In both cases:

1. If $a > 0$, the parabola "opens upward."
2. If $a < 0$, the parabola "opens downward."

Conic Sections

Equations of a Horizontal Parabola

$x = ay^2 + by + c$ or $x = a(y-k)^2 + h$ where $a \neq 0$.

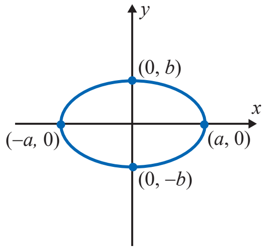
The parabola opens left if $a < 0$ and right if $a > 0$.

The vertex is at (h, k).

The line $y = k$ is the line of symmetry.

Equation of an Ellipse

The standard form for the equation of an ellipse with its center at the origin is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



The points $(a, 0)$ and $(-a, 0)$ are the x-intercepts (called vertices).

The points $(0, b)$ and $(0, -b)$ are the y-intercepts (called vertices).

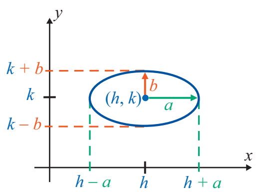
When $a^2 > b^2$:

- The segment of length $2a$ joining the x-intercepts is called the major axis.
- The segment of length $2b$ joining the y-intercepts is called the minor axis.

When $b^2 > a^2$:

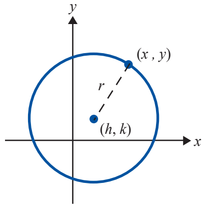
- The segment of length $2b$ joining the y-intercepts is called the major axis.
- The segment of length $2a$ joining the x-intercepts is called the minor axis.

The standard form for the equation of an ellipse with its center at (h, k) is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.



Equation of a Circle

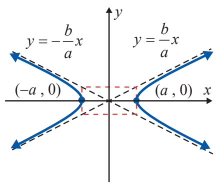
The equation of a circle with radius r and center (h, k) is $(x-h)^2 + (y-k)^2 = r^2$.



Equation of a Hyperbola

In general, there are two standard forms for equations of hyperbolas with their centers at the origin.

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



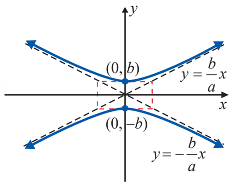
x-intercepts (vertices) at $(a, 0)$ and $(-a, 0)$

No y-intercepts

Asymptotes: $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$

The curves "open" left and right.

2. $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$



y-intercepts (vertices) at $(0, b)$ and $(0, -b)$

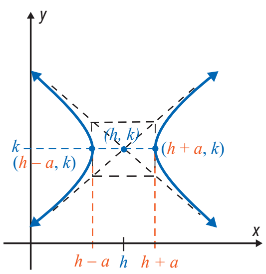
No x-intercepts

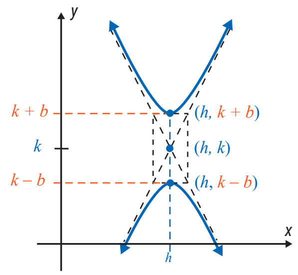
Asymptotes: $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$

The curves "open" up and down.

The equation of a hyperbola with its center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$





Sequences and Series

Sequences

An **infinite sequence** (or a **sequence**) is a function that has the positive integers as its domain.

An **alternating sequence** is a sequence in which the terms alternate in sign.

A sequence is **decreasing** if successive terms become smaller.

A sequence is **increasing** if successive terms become larger.

Properties of Sigma Notation

For sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, the following are true.

I. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for any i, $1 \le i \le n-1$

II. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

III. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

IV. $\sum_{k=1}^{n} c = nc$

Arithmetic Sequences

A sequence $\{a_n\}$ is an **arithmetic sequence** if successive terms have a common difference, d.

$$a_{k+1} - a_k = d$$

The general n^{th} term has the form $a_n = a_1 + (n-1)d$.

The n^{th} partial sum S_n is $S_n = \sum_{k=1}^{n} a_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series

A sequence $\{a_n\}$ is a **geometric sequence** if successive terms have a common ratio, r.

$$\frac{a_{k+1}}{a_k} = r$$

The general n^{th} term has the form $a_n = a_1 r^{n-1}$.

The n^{th} partial sum S_n is $S_n = \sum_{k=1}^{n} a_k = \frac{a_1(1-r^n)}{1-r}$.

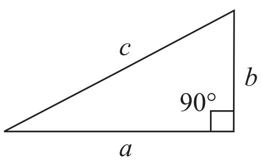
The indicated sum of all terms of a sequence is called an **infinite series** (or a series).

The sum of the infinite geometric series is $S = \sum_{k=1}^{\infty} a_k = \frac{a_1}{1-r}$.

Formulas and Theorems

The Pythagorean Theorem

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs. $c^2 = a^2 + b^2$



Special Products

1. $x^2 - a^2 = (x+a)(x-a)$ Difference of two squares
2. $x^2 + 2ax + a^2 = (x+a)^2$ Square of a binomial sum
3. $x^2 - 2ax + a^2 = (x-a)^2$ Square of a binomial difference
4. $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$ Sum of two cubes
5. $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$ Difference of two cubes

Change-of-Base Formula for Logarithms

For $a, b, x > 0$ and $a, b \ne 1$, $\log_b x = \frac{\log_a x}{\log_a b}$.

Distance Between Two Points

The distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Midpoint Formula

The midpoint between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The Binomial Theorem

For real numbers a and b and a nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Binomial Coefficient $\binom{n}{r}$

For nonnegative integers n and r, with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Matrices and Determinants

Matrices

System of Linear Equations	Coefficient Matrix	Augmented Matrix
$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\left[\begin{array}{cc:c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Multiply a row by a nonzero constant and add it to another row.

Upper Triangular Form and Row Echelon Form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

A matrix is in **upper triangular form** if its entries in the lower left triangular region are all 0's. If a_{11}, a_{22}, and a_{33} (the entries along the main diagonal) all equal 1 when the matrix is in upper triangular form, then the matrix is also in **row echelon form** (or **ref**).

Gaussian Elimination

1. Write the augmented matrix for the system.
2. Use elementary row operations to transform the matrix into row echelon form.
3. Solve the corresponding system of equations by using back substitution.

Determinants

Value of a 2 × 2 Determinant:

For the square matrix, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Value of a 3 × 3 Determinant:

For the square matrix, $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(\text{minor of } a_{11}) - a_{12}(\text{minor of } a_{12}) + a_{13}(\text{minor of } a_{13})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cramer's Rule

For the system $\begin{cases} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{cases}$,

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_x = \begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}, \quad \text{and} \quad D_y = \begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix},$$

if $D \neq 0$, then $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$ is the unique solution to the system.