

Table of Integrals

With the exception of exponent restrictions, antiderivatives in this table appear without conditions—some are valid only when appropriate restrictions on parameters (e.g., $a > 0$) and/or variables (e.g., $|x| > 1$) are applied.

Elementary Integrals

1. $\int k \, dx = kx + C$
2. $\int x^r \, dx = \frac{x^{r+1}}{r+1} + C, \quad r \neq -1$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x \, dx = e^x + C$
5. $\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$
6. $\int \sin x \, dx = -\cos x + C$
7. $\int \cos x \, dx = \sin x + C$
8. $\int \sec^2 x \, dx = \tan x + C$
9. $\int \csc^2 x \, dx = -\cot x + C$
10. $\int \sec x \tan x \, dx = \sec x + C$
11. $\int \csc x \cot x \, dx = -\csc x + C$
12. $\int \tan x \, dx = \ln|\sec x| + C$
13. $\int \cot x \, dx = \ln|\sin x| + C$
14. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

Integrals with Expressions of the Form $a + bx$

16. $\int \frac{dx}{a+bx} = \frac{1}{b} \ln|a+bx| + C$
17. $\int \frac{x}{a+bx} \, dx = \frac{1}{b^2} (bx - a \ln|a+bx|) + C$
18. $\int \frac{x^2}{a+bx} \, dx = \frac{1}{2b^3} (b^2 x^2 - 2abx + 2a^2 \ln|a+bx|) + C$
19. $\int \frac{dx}{x(a+bx)} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right| + C$
20. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} - \frac{b}{a^2} \ln \left| \frac{x}{a+bx} \right| + C$
21. $\int \frac{x}{(a+bx)^2} \, dx = \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln|a+bx| \right) + C$
22. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2} \ln \left| \frac{x}{a+bx} \right| + C$
23. $\int \frac{x^2}{(a+bx)^2} \, dx = \frac{1}{b^3} \left(bx - \frac{a^2}{a+bx} - 2a \ln|a+bx| \right) + C$
24. $\int x\sqrt{a+bx} \, dx = \frac{2}{15b^2} (3bx - 2a)(a+bx)^{3/2} + C$
25. $\int \frac{x}{\sqrt{a+bx}} \, dx = \frac{2}{3b^2} (bx - 2a)\sqrt{a+bx} + C$
26. $\int \frac{x^2}{\sqrt{a+bx}} \, dx = \frac{2}{15b^3} (8a^2 + 3b^2 x^2 - 4abx)\sqrt{a+bx} + C$
27. $\int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} -\frac{2}{\sqrt{a}} \tanh^{-1} \sqrt{\frac{a+bx}{a}} + C = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right| + C & \text{if } a > 0 \\ \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}} + C & \text{if } a < 0 \end{cases}$

28. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}}$
29. $\int \frac{\sqrt{a+bx}}{x^2} dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{a+bx}}$
30. $\int x^n \sqrt{a+bx} dx = \frac{2}{b(2n+3)} \left[x^n (a+bx)^{3/2} - na \int x^{n-1} \sqrt{a+bx} dx \right]$
31. $\int \frac{x^n}{\sqrt{a+bx}} dx = \frac{2x^n \sqrt{a+bx}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{x^{n-1}}{\sqrt{a+bx}} dx$
32. $\int \frac{dx}{x^n \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{a(n-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{dx}{x^{n-1} \sqrt{a+bx}}$

Integrals with Expressions of the Form $a^2 + x^2$

33. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
34. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
35. $\int x^2 \sqrt{a^2 + x^2} dx = \frac{x}{8} (a^2 + 2x^2) \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
36. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$
37. $\int \frac{\sqrt{a^2 + x^2}}{x^2} dx = -\frac{\sqrt{a^2 + x^2}}{x} + \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
38. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right) + C = \sinh^{-1} \left(\frac{x}{a} \right) + C$
39. $\int \frac{x^2}{\sqrt{a^2 + x^2}} dx = \frac{x}{2} \sqrt{a^2 + x^2} - \frac{a^2}{2} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
40. $\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \ln \left| \frac{\sqrt{a^2 + x^2} + a}{x} \right| + C = -\frac{1}{a} \operatorname{csch}^{-1} \left| \frac{x}{a} \right| + C$
41. $\int \frac{dx}{x^2 \sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C$
42. $\int (a^2 + x^2)^{3/2} dx = \frac{x}{8} (2x^2 + 5a^2) \sqrt{a^2 + x^2} + \frac{3a^4}{8} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$
43. $\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C$

Integrals with Expressions of the Form $a^2 - x^2$

$$44. \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C = \begin{cases} \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + C & \text{if } x^2 < a^2 \\ \frac{1}{a} \coth^{-1} \left(\frac{x}{a} \right) + C & \text{if } x^2 > a^2 \end{cases}$$

$$45. \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$46. \int x^2 \sqrt{a^2 - x^2} \, dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$47. \int \frac{\sqrt{a^2 - x^2}}{x} \, dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C$$

$$48. \int \frac{\sqrt{a^2 - x^2}}{x^2} \, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$49. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$50. \int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$51. \int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C = -\frac{1}{a} \operatorname{sech}^{-1} \left(\frac{x}{a} \right) + C$$

$$52. \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{1}{a^2 x} \sqrt{a^2 - x^2} + C$$

$$53. \int (a^2 - x^2)^{3/2} \, dx = -\frac{x}{8} (2x^2 - 5a^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$54. \int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C$$

Integrals with Expressions of the Form $x^2 - a^2$

$$55. \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$56. \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$57. \int x^2 \sqrt{x^2 - a^2} \, dx = \frac{x}{8} (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$58. \int \frac{\sqrt{x^2 - a^2}}{x} \, dx = \sqrt{x^2 - a^2} - a \sec^{-1} \left| \frac{x}{a} \right| + C$$

$$59. \int \frac{\sqrt{x^2 - a^2}}{x^2} dx = -\frac{\sqrt{x^2 - a^2}}{x} + \ln|x + \sqrt{x^2 - a^2}| + C$$

$$60. \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$61. \int \frac{x^2}{\sqrt{x^2 - a^2}} dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln|x + \sqrt{x^2 - a^2}| + C$$

$$62. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$63. \int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$$

$$64. \int (x^2 - a^2)^{3/2} dx = \frac{x}{8}(2x^2 - 5a^2)\sqrt{x^2 - a^2} + \frac{3a^4}{8}\ln|x + \sqrt{x^2 - a^2}| + C$$

$$65. \int \frac{dx}{(x^2 - a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2 - a^2}} + C$$

Integrals with Trigonometric Expressions

$$66. \int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$67. \int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$68. \int \tan^2 x dx = \tan x - x + C$$

$$69. \int \cot^2 x dx = -\cot x - x + C$$

$$70. \int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$71. \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$72. \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

$$73. \int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$$

$$74. \int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

$$75. \int \csc^n x dx = -\frac{\cot x \csc^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx$$

$$76. \int \sin(ax)\sin(bx) dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} + C$$

$$77. \int \cos(ax)\cos(bx) dx = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} + C$$

$$78. \int \sin(ax)\cos(bx) dx = -\frac{\cos((a-b)x)}{2(a-b)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

$$79. \int x \sin x dx = \sin x - x \cos x + C$$

$$80. \int x \cos x dx = \cos x + x \sin x + C$$

$$81. \int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx$$

$$82. \int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$$

$$\begin{aligned}
 83. \quad \int \sin^m x \cos^n x \, dx &= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+m} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx \\
 &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx
 \end{aligned}$$

Integrals with Inverse Trigonometric Expressions

$$\begin{aligned}
 84. \quad \int \sin^{-1} x \, dx &= x \sin^{-1} x + \sqrt{1-x^2} + C & 85. \quad \int \cos^{-1} x \, dx &= x \cos^{-1} x - \sqrt{1-x^2} + C \\
 86. \quad \int \tan^{-1} x \, dx &= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C & 87. \quad \int \cot^{-1} x \, dx &= x \cot^{-1} x + \frac{1}{2} \ln(x^2+1) + C \\
 88. \quad \int \sec^{-1} x \, dx &= x \sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C & 89. \quad \int \csc^{-1} x \, dx &= x \csc^{-1} x + \ln|x + \sqrt{x^2-1}| + C \\
 90. \quad \int x^n \sin^{-1} x \, dx &= \frac{1}{n+1} \left(x^{n+1} \sin^{-1} x - \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx \right), \quad n \neq -1 \\
 91. \quad \int x^n \cos^{-1} x \, dx &= \frac{1}{n+1} \left(x^{n+1} \cos^{-1} x + \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx \right), \quad n \neq -1 \\
 92. \quad \int x^n \tan^{-1} x \, dx &= \frac{1}{n+1} \left(x^{n+1} \tan^{-1} x - \int \frac{x^{n+1}}{1+x^2} dx \right), \quad n \neq -1
 \end{aligned}$$

Integrals with Exponential or Logarithmic Expressions

$$\begin{aligned}
 93. \quad \int x e^{ax} \, dx &= \frac{1}{a^2} (ax-1) e^{ax} + C & 94. \quad \int x^n e^{ax} \, dx &= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \\
 95. \quad \int \sin(ax) e^{bx} \, dx &= \frac{b \sin(ax) - a \cos(ax)}{a^2 + b^2} e^{bx} + C & 96. \quad \int \cos(ax) e^{bx} \, dx &= \frac{b \cos(ax) + a \sin(ax)}{a^2 + b^2} e^{bx} + C \\
 97. \quad \int \ln x \, dx &= x \ln x - x + C & 98. \quad \int (\ln x)^n \, dx &= x (\ln x)^n - n \int (\ln x)^{n-1} \, dx \\
 99. \quad \int x^n \ln x \, dx &= \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C & 100. \quad \int \frac{dx}{x \ln x} &= \ln |\ln x| + C
 \end{aligned}$$

Integrals with Expressions of the Form $\sqrt{2ax-x^2}$

$$\begin{aligned}
 101. \quad \int \sqrt{2ax-x^2} \, dx &= \frac{x-a}{2} \sqrt{2ax-x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C \\
 102. \quad \int x \sqrt{2ax-x^2} \, dx &= \frac{2x^2-ax-3a^2}{6} \sqrt{2ax-x^2} + \frac{a^3}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C \\
 103. \quad \int \frac{\sqrt{2ax-x^2}}{x} \, dx &= \sqrt{2ax-x^2} + a \sin^{-1} \left(\frac{x-a}{a} \right) + C
 \end{aligned}$$

104. $\int \frac{\sqrt{2ax-x^2}}{x^2} dx = -\frac{2}{x}\sqrt{2ax-x^2} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$
105. $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$
106. $\int \frac{x}{\sqrt{2ax-x^2}} dx = -\sqrt{2ax-x^2} + a \sin^{-1}\left(\frac{x-a}{a}\right) + C$
107. $\int \frac{x^2}{\sqrt{2ax-x^2}} dx = -\frac{x+3a}{2}\sqrt{2ax-x^2} + \frac{3a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$
108. $\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{ax}\sqrt{2ax-x^2} + C$

Integrals with Hyperbolic Expressions

109. $\int \sinh x dx = \cosh x + C$
110. $\int \cosh x dx = \sinh x + C$
111. $\int \tanh x dx = \ln \cosh x + C$
112. $\int \coth x dx = \ln |\sinh x| + C$
113. $\int \operatorname{sech} x dx = \tan^{-1} |\sinh x| + C$
114. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right| + C$
115. $\int \operatorname{sech}^2 x dx = \tanh x + C$
116. $\int \operatorname{csch}^2 x dx = -\coth x + C$
117. $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$
118. $\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$

Integrals with Inverse Hyperbolic Expressions

119. $\int \sinh^{-1} x dx = x \sinh^{-1} x - \sqrt{1+x^2} + C$
120. $\int \cosh^{-1} x dx = x \cosh^{-1} x - \sqrt{x^2-1} + C$
121. $\int \tanh^{-1} x dx = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) + C$
122. $\int \coth^{-1} x dx = x \coth^{-1} x + \frac{1}{2} \ln(1-x^2) + C$
123. $\int \operatorname{sech}^{-1} x dx = x \operatorname{sech}^{-1} x + 2 \sin^{-1} \sqrt{\frac{1+x}{2}} + C$
124. $\int \operatorname{csch}^{-1} x dx = \begin{cases} x \operatorname{csch}^{-1} x + \sinh^{-1} x + C & \text{if } x > 0 \\ x \operatorname{csch}^{-1} x - \sinh^{-1} x + C & \text{if } x < 0 \end{cases}$

Selected Definite Integrals

125. $\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$
126. $\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$
127. $\int_0^\infty e^{-ax} \cos(bx) dx = \frac{a}{a^2 + b^2}$
128. $\int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2 + b^2}$

ALGEBRA

Properties of Absolute Value

For all real numbers a and b :

$$|a| \geq 0$$

$$|-a| = |a|$$

$$a \leq |a|$$

$$|ab| = |a||b|$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}, \quad b \neq 0$$

$$|a + b| \leq |a| + |b| \quad \text{Triangle Inequality}$$

Properties of Integer Exponents and Radicals

Assume that n and m are positive integers, that a and b are nonnegative, and that all denominators are nonzero. See Appendices B and D for graphs and further discussion.

$$a^n \cdot a^m = a^{n+m}$$

$$(a^n)^m = a^{nm}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$(ab)^n = a^n b^n$$

$$a^{-n} = \frac{1}{a^n}$$

$$\left(\frac{a}{b} \right)^n = \frac{a^n}{b^n}$$

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a} \right)^m$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Special Product Formulas

$$(A - B)(A + B) = A^2 - B^2$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

Factoring Special Binomials

$$A^2 - B^2 = (A - B)(A + B)$$

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Trigonometric and Hyperbolic Functions: Definitions, Graphs, and Identities

See Appendix C.

Distance Formula

The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Slope of a Line

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \begin{array}{l} \text{Horizontal lines } y = c \text{ have slope } 0. \\ \text{Vertical lines } x = c \text{ have undefined slope.} \end{array}$$

Parallel and Perpendicular Lines

Given a line with slope m :

slope of parallel line = m

slope of perpendicular line = $-1/m$

Forms of Equations of a Line

Standard Form: $ax + by = c$

Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y -intercept

Point-Slope Form: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line

Vector Form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is a fixed vector and $\mathbf{v}$ is a direction vector

Properties of Logarithms

Let a , b , x , and y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number. See Appendix B for graphs and further discussion.

$\log_a x = y$ and $x = a^y$ are equivalent

$$\log_a 1 = 0 \quad \log_a a = 1$$

$$\log_a (a^x) = x \quad a^{\log_a x} = x$$

$$\log_a (xy) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

$$\log_a (x^r) = r \log_a x$$

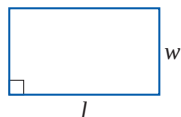
$$\log_b x = \frac{\log_a x}{\log_a b} \quad \text{Change of base formula}$$

GEOMETRY

A = area, C = circumference, SA = surface area or lateral area, V = volume

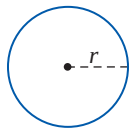
Rectangle

$$A = lw$$



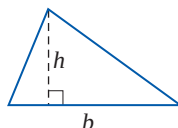
Circle

$$A = \pi r^2 \quad C = 2\pi r$$



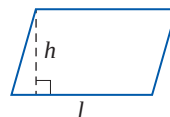
Triangle

$$A = \frac{1}{2}bh$$



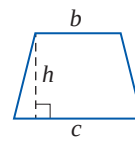
Parallelogram

$$A = lh$$



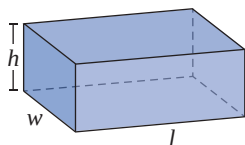
Trapezoid

$$A = \frac{1}{2}h(b+c)$$



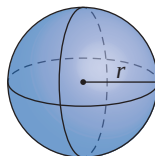
Rectangular Prism

$$V = lwh \quad SA = 2lh + 2wh + 2lw$$



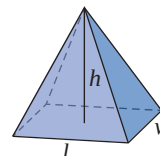
Sphere

$$V = \frac{4}{3}\pi r^3 \quad SA = 4\pi r^2$$



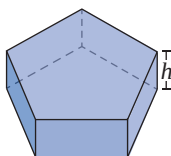
Rectangular Pyramid

$$V = \frac{1}{3}lwh$$



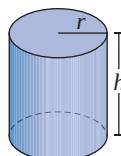
Right Cylinder

$$V = (\text{Area of Base})h$$



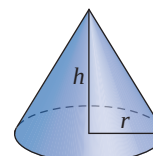
Right Circular Cylinder

$$V = \pi r^2 h \quad SA = 2\pi r^2 + 2\pi rh$$



Cone

$$V = \frac{1}{3}\pi r^2 h \quad SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$



LIMITS

Definition of Limit

Let f be a function defined on an open interval containing c , except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is L , and write $\lim_{x \rightarrow c} f(x) = L$, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever x satisfies $0 < |x - c| < \delta$.

Basic Limit Laws

Sum/Difference Law: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x)$

Constant Multiple Law: $\lim_{x \rightarrow c} [kf(x)] = k \lim_{x \rightarrow c} f(x)$

Product Law: $\lim_{x \rightarrow c} [f(x)g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$

Quotient Law: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$, provided $\lim_{x \rightarrow c} g(x) \neq 0$

Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at c itself, and if $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$ as well.

Continuity at a Point

Given a function f defined on an open interval containing c , we say f is continuous at c if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

L'Hôpital's Rule

Suppose f and g are differentiable at all points of an open interval I containing c , and that $g'(x) \neq 0$ for all $x \in I$ except possibly at $x = c$. Suppose further that either

$$\lim_{x \rightarrow c} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = 0$$

or

$$\lim_{x \rightarrow c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = \pm\infty.$$

Then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right is a real number or ∞ or $-\infty$.

DERIVATIVES

The Derivative of a Function

The derivative of f , denoted f' , is the function whose value at the point x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Elementary Differentiation Rules

Constant Rule: $\frac{d}{dx}(k) = 0$

Constant Multiple Rule: $\frac{d}{dx}[kf(x)] = kf'(x)$

Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Power Rule: $\frac{d}{dx}(x^r) = rx^{r-1}$

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b) , then there is at least one point $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad |x| < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, \quad |x| > 1$$

The Derivative Rule for Inverse Functions

If a function f is differentiable on an interval (a, b) , and if $f'(x) \neq 0$ for all $x \in (a, b)$, then f^{-1} both exists and is differentiable on the image of the interval (a, b) under f , denoted as $f((a, b))$ in the formula below. Further,

$$\text{if } x \in (a, b), \text{ then } (f^{-1})'(f(x)) = \frac{1}{f'(x)},$$

and

$$\text{if } x \in f((a, b)), \text{ then } (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

INTEGRATION

Properties of the Definite Integral

Given the integrable functions f and g on the interval $[a, b]$ and any constant k , the following properties hold.

1. $\int_a^a f(x) dx = 0$
2. $\int_b^a f(x) dx = -\int_a^b f(x) dx$
3. $\int_a^b k dx = k(b-a)$
4. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
5. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
6. $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$, assuming each integral exists
7. If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.
8. If $m = \min_{a \leq x \leq b} f(x)$ and $M = \max_{a \leq x \leq b} f(x)$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$.

The Fundamental Theorem of Calculus

Part I

Given a continuous function f on an interval I and a fixed point $a \in I$, define the function F on I by $F(x) = \int_a^x f(t) dt$. Then $F'(x) = f(x)$ for all $x \in I$.

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is the interval I , and if f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

Hence, if F is an antiderivative of f on I ,

$$\int f(g(x))g'(x) dx = F(g(x)) + C.$$

Integration by Parts

Given differentiable functions f and g ,

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

If we let $u = f(x)$ and $v = g(x)$, then $du = f'(x)dx$ and $dv = g'(x)dx$ and the equation takes on the more easily remembered differential form

$$\int u dv = uv - \int v du.$$

Part II

If f is a continuous function on the interval $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

SEQUENCES AND SERIES

Summation Facts and Formulas

Constant Rule for Finite Sums:

$$\sum_{i=1}^n c = nc, \text{ for any constant } c$$

Constant Multiple Rule for Finite Sums:

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i, \text{ for any constant } c$$

Sum/Difference Rule for Finite Sums:

$$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

Sum of the First n Positive Integers:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Sum of the First n Squares:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of the First n Cubes:

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Series

For a geometric sequence $\{a_n\}$ with common ratio r :

$$\text{Partial Sum: } s_n = \frac{a(1-r^n)}{1-r}, \text{ if } |r| \neq 1$$

$$\text{Infinite Sum: } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \text{ if } |r| < 1$$

Binomial Series

For any real number m and $-1 < x < 1$:

$$\begin{aligned} (1+x)^m &= \sum_{n=0}^{\infty} \binom{m}{n} x^n \\ &= 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots \\ &\quad + \frac{m(m-1)\dots(m-n+1)}{n!} x^n + \dots \end{aligned}$$

Taylor Series and Maclaurin Series

Given a function f with derivatives of all orders throughout an open interval containing a , the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

is called the Taylor series generated by f about a . The Taylor series generated by f about 0 is also known as the Maclaurin series generated by f .

VECTOR CALCULUS

Properties of Scalar Multiplication and Vector Addition

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and scalars a and b :

Scalar Multiplication Properties **Vector Addition Properties**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$$

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

$$(ab)\mathbf{u} = a(b\mathbf{u}) = b(a\mathbf{u})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}, 0\mathbf{u} = \mathbf{0}, \text{ and } a\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

$$|a\mathbf{u}| = |a||\mathbf{u}|$$

Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, the dot product $\mathbf{u} \cdot \mathbf{v}$ of the two vectors is the scalar defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar formula defines the dot product of two vectors in $\mathbb{R}^2$.

Properties of the Dot Product

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and scalar a :

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$\mathbf{0} \cdot \mathbf{u} = 0$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

$$a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v})$$

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$$

Dot Product and the Angle between Two Vectors

If two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are depicted so that their initial points coincide, and if θ represents the smaller of the two angles formed by $\mathbf{u}$ and $\mathbf{v}$ (so that $0 \leq \theta \leq \pi$), then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

Projection of $\mathbf{u}$ onto $\mathbf{v}$

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

Cross Product

Given $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

Properties of the Cross Product

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ and scalars a and b :

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$\mathbf{0} \times \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

$$(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Coordinate Conversion Relationships

Cylindrical and
Cartesian

$$r^2 = x^2 + y^2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Spherical and
Cylindrical

$$\rho^2 = r^2 + z^2$$

$$r = \rho \sin \varphi$$

$$\theta = \theta$$

$$z = \rho \cos \varphi$$

Spherical and
Cartesian

$$\rho^2 = x^2 + y^2 + z^2$$

$$x = \underbrace{\rho \sin \varphi}_{r} \cos \theta$$

$$y = \underbrace{\rho \sin \varphi}_{r} \sin \theta$$

$$z = \rho \cos \varphi$$

Gradient Vector

Given a function $f(x_1, x_2, \dots, x_n)$,

$$\nabla f(x_1, x_2, \dots, x_n)$$

$$= \left\langle f_{x_1}(x_1, x_2, \dots, x_n), f_{x_2}(x_1, x_2, \dots, x_n), \dots, f_{x_n}(x_1, x_2, \dots, x_n) \right\rangle.$$

Computation of $D_{\mathbf{u}}f(\mathbf{c})$ (Directional Derivative)

Assuming the derivative of $f(x, y)$ at the point (a, b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ exists,

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle = \nabla f(a, b) \cdot \mathbf{u}.$$

More generally, if $f(x_1, x_2, \dots, x_n)$ is differentiable at the point $\mathbf{c} = (c_1, c_2, \dots, c_n)$ and if $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ is a unit vector, then

$$D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{u}.$$

Properties of the Gradient

Assume f and g are both differentiable functions and that k is a fixed real number. Then the following laws hold.

Sum/Difference Law: $\nabla(f \pm g) = \nabla f \pm \nabla g$

Constant Multiple Law: $\nabla(kf) = k\nabla f$

Product Law: $\nabla(fg) = f\nabla g + g\nabla f$

Quotient Law: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, provided $g \neq 0$

The Fundamental Theorem for Line Integrals (Gradient Theorem)

Assume that f is a differentiable function whose gradient ∇f is continuous along a curve C and that C is defined by the smooth vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Divergence (Flux Density) of a Vector Field

The divergence, or flux density, of a vector field

$\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ is the scalar function

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

In general, we can denote the divergence of a vector field $\mathbf{F}$ as the dot product of the del operator and $\mathbf{F}$.

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

Curl of a Vector Field

The curl of a vector field

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

is the vector function

$$\operatorname{curl} \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

The curl of a vector field $\mathbf{F}$ can be remembered as the cross product of the del operator and $\mathbf{F}$.

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$

Green's Theorem (Tangential-Curl Form)

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let R be the region enclosed by C . If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ and P and Q have continuous partial derivatives on an open region containing R , then

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{T} ds &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_C P dx + Q dy \\ &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \end{aligned}$$

Extending $\mathbf{F}$ to $\mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle$ and using the fact that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \nabla \times \mathbf{F} \cdot \mathbf{k},$$

we can write this version of the formula as

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA.$$

Green's Theorem (Normal-Divergence Form)

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let R be the region enclosed by C . If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ and P and Q have continuous partial derivatives on an open region containing R , then

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n} ds &= \oint_C P dy - Q dx \\ &= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \\ &= \iint_R \nabla \cdot \mathbf{F} dA. \end{aligned}$$

Stokes' Theorem

Assume $\mathbf{F}$ is a vector field with continuous partial derivatives in an open region of space containing a piecewise smooth surface S . Assume that the boundary of S is a simple, closed, piecewise smooth curve C , and that C is positively oriented with respect to S . Then

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma.$$

The Divergence Theorem

Assume $\mathbf{F}$ is a vector field with continuous partial derivatives in an open region of space containing $D \subseteq \mathbb{R}^3$, and assume D is either a simple union or a finite union of simple regions. Let S denote the surface of D , and let $\mathbf{n}$ be the outward-pointing field of unit vectors normal to S . Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV.$$